

T-COMITANTS AND THE PROBLEM OF A CENTER FOR QUADRATIC DIFFERENTIAL SYSTEMS

A.M.VOLDMAN, N.I.VULPE

ABSTRACT.

The new necessary and sufficient affine invariant conditions for the existence and for determining the number of centers for general quadratic system are pointed out. These conditions correspond to the partition of 12-dimensional coefficient space of indicated system with respect to the number and the multiplicity of its finite critical points.

Let us consider system of differential equations

$$\begin{aligned} \frac{dx^1}{dt} &= a^1 + a_\alpha^1 x^\alpha + a_{\alpha\beta}^1 x^\alpha x^\beta \equiv P_0 + P_1 + P_2, \\ \frac{dx^2}{dt} &= a^2 + a_\alpha^2 x^\alpha + a_{\alpha\beta}^2 x^\alpha x^\beta \equiv Q_0 + Q_1 + Q_2, \end{aligned} \quad (j, \alpha, \beta = 1, 2) \quad (1)$$

where a_α^j and $a_{\alpha\beta}^j$ ($j, \alpha, \beta = 1, 2$) are real numbers (the tensor $a_{\alpha\beta}^j$ is symmetric in the lower indices, with respect to which the complete contraction was made) and $P_i(x^1, x^2)$ ($i = 0, 1, 2$) are homogeneous polynomials of degree i .

The existence of a center at the origin for system (1) was examined for the first time in 1904 by H.Dulac [1]. More precisely, in [1] the problem of a center was considered for the following equation:

$$\frac{dy}{dx} = -\frac{x + ax^2 + bxy + cy^2}{y + a'x^2 + b'xy + c'y^2}, \quad (2)$$

with the complex variables x and y and the complex coefficients a, b, c, a', b', c' .

In W.Kapteyn's papers [2,3] the analogical problem was examined for the equation (2) with real variables and coefficients.

However, neither Dulac nor Kapteyn have been obtained explicit conditions, through the coefficients of equation (2), which ensure the existence of a center at the origin. This problem, besides the determination of the qualitative phase portraits of equation (2) with a center, were stated by M. Frommer in [4]. But, as was shown by N.A.Saharnicov [5], there are some mistakes in Frommer's paper.

K.S.Sibirsky [6] and L.N.Belyustina [7] are the first who found out the explicit conditions, expressed through the coefficients of equation (2) for the existence of a center. We remark, that the center problem for equation (2) was also examined by K.E.Malkin [8], I.S.Kukles [9] and other authors.

Some years later, applying the developed theory of algebraic invariants of differential equations, K.S.Sibirsky [10] solved the problem under consideration for

a more general case. The center affine invariant conditions for the existence of a center at the origin are determined for system (1), i.e. for system with $a^1 = a^2 = 0$.

Finally, the necessary and sufficient affine invariant conditions for the existence and for determining the number of centers (anywhere in the plane) for a general system (1) were established in [11,12]. The obtained conditions are expressed as equalities or inequalities involving polynomials, with the degrees at most 24. The following question was formulated:

QUESTION [11]. *Is there a semialgebraic solution of this problem with a lower maximal degree for the polynomials ?*

In this paper, by using T-comitants, we have answered this question affirmatively. We have pointed out the new necessary and sufficient affine invariant conditions for the existence and for determining the number of centers for system (1), corresponding to the partition of the coefficient space R^{12} of a non-degenerated system (1) with respect to the number and the multiplicity of the finite critical points of this system [13,14].

PRELIMINARIES

Let $a \in R^{12}$ be an element of the space of the coefficients of the system (1) and let us consider the group Q of nondegenerate real linear transformations of the phase plane. We denote by r_q the linear presentation of any element $q \in Q$ into the coefficient space R^{12} of system (1).

Definition 1. [15] A polynomial $K(a, x)$ of the coefficients of system (1) and the unknown variables x^1 and x^2 is called a comitant of system (1) in the group Q , if there exists a function $\lambda(q)$ such that

$$K(r_q \cdot a, q \cdot x) \equiv \lambda(q)K(a, x)$$

for every $q \in Q$, $a \in R^{12}$ and $x = (x^1, x^2)$.

A comitant K of system (1) in the group $Q = GL(2, R)$ of linear homogeneous transformations of the phase plane of system (1) (which is also called a group of center-affine transformations) is called center affine. A comitant K of the system (1) in the group $Q = Aff(2, R)$ of affine (linear non-homogenous) transformations is called affine. If the comitant K does not depend explicitly on the variables x^1 and x^2 then it is called an invariant (center affine or affine, respectively).

Remark 1. We say that the comitant of system (1) equals zero when all its coefficients vanish. The signs of the comitants which take part in some sequences of conditions should be calculated at one and the same point, where they do not vanish.

We denote by $T(2, R)$ the group of shift transformations and by r_t the linear presentation of any element $t \in T$ into the coefficient space R^{12} of system (1).

Definition 2. [16] A comitant $K(a, x)$ of system (1) is called a T-comitant if the relation

$$K(r_t \cdot a, x) \equiv K(a, x)$$

is valid for every $t \in T$ and $a \in R^{12}$.

Definition 3. [17] *The polynomial*

$$(f, \varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!} \sum_{h=0}^k (-1)^h C_k^h \frac{\partial^k f}{\partial(x^1)^{k-h} \partial(x^2)^h} \frac{\partial^k \varphi}{\partial(x^1)^k \partial(x^2)^{k-h}}$$

is called a transvectant of index k of two forms f and φ . The degree of these forms in the coordinates of the vector $x = (x^1, x^2)$ are equal to r and ρ , respectively and $k \leq \min(r, \rho)$.

Proposition 1. [16] *The transvectant $(f, \varphi)^{(k)}$ of two T -comitants f and φ is also a T -comitant.*

According to [16], by using the following T -comitants

$$\begin{aligned} \hat{A} &= a_k^p a_{\alpha m}^q a_{ln}^\alpha \varepsilon_{pq} \varepsilon^{kl} \varepsilon^{mn}, \\ \hat{B} &= [2a_u^n a_{u\alpha}^h - a_u^n a_\alpha^h] a_r^l a_{p\beta}^k a_{qs}^m a_{v\gamma}^g x^\alpha x^\beta x^\gamma \varepsilon_{kl} \varepsilon_{mn} \varepsilon_{gh} \varepsilon^{pq} \varepsilon^{rs} \varepsilon^{uv}, \\ \hat{C} &= a_{\alpha\beta}^p x^q x^\alpha x^\beta \varepsilon_{pq}, \\ \hat{D} &= [2a_\alpha^p a_{\gamma\gamma}^r - a_\alpha^p a_\gamma^r] a_{\beta\gamma}^u x^q x^s x^v \varepsilon_{pq} \varepsilon_{rs} \varepsilon_{uv} \varepsilon^{\alpha\beta} \varepsilon^{\gamma\gamma}, \\ \hat{E} &= a_k^p a_{\alpha m}^q a_{ln}^r x^\alpha \varepsilon_{pq} \varepsilon_{rs} \varepsilon^{kl} \varepsilon^{mn}, \\ \hat{F} &= [a_s^m a_\beta^n a_{pr}^k - 2a_r^k a_\beta^n a_{ps}^m + a_p^k a_r^m a_{s\beta}^n - 4a_m^m a_{pr}^k a_{s\beta}^n] a_{q\alpha}^l x^\alpha x^\beta \varepsilon_{kl} \varepsilon_{mn} \varepsilon^{pq} \varepsilon^{rs}, \\ \hat{G} &= a_{\alpha\beta}^\alpha x^\beta, \\ \hat{H} &= \frac{1}{2} a_{r\alpha}^p a_{s\beta}^q x^\alpha x^\beta \varepsilon_{pq} \varepsilon^{rs}, \\ \hat{K} &= \frac{1}{2} a_{mu}^p a_{nv}^r x^q x^s \varepsilon_{pq} \varepsilon_{rs} \varepsilon^{mn} \varepsilon^{uv} \end{aligned}$$

(where $\varepsilon^{11} = \varepsilon^{22} = \varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21} = -\varepsilon_{21} = 1$), in virtue of Proposition 1, the following affine invariants can be constructed:

$$\begin{aligned} A_1 &= \hat{A}, \quad A_2 = (\hat{C}, \hat{D})^{(3)}, \quad A_3 = (((\hat{C}, \hat{G})^{(1)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \\ A_4 &= (\hat{H}, \hat{H})^{(2)}, \quad A_5 = (\hat{H}, \hat{K})^{(2)}, \quad A_6 = (\hat{E}, \hat{H})^{(2)}, \\ A_7 &= ((\hat{C}, \hat{E})^{(2)}, \hat{G})^{(1)}, \quad A_8 = ((\hat{D}, \hat{H})^{(2)}, \hat{G})^{(1)}, \\ A_9 &= (((\hat{D}, \hat{G})^{(1)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \quad A_{10} = ((\hat{D}, \hat{K})^{(2)}, \hat{G})^{(1)}, \\ A_{11} &= (\hat{F}, \hat{K})^{(2)}, \quad A_{12} = (\hat{F}, \hat{H})^{(2)}, \quad A_{13} = (((\hat{C}, \hat{H})^{(1)}, \hat{H})^{(2)}, \hat{G})^{(1)}, \\ A_{14} &= (\hat{B}, \hat{C})^{(3)}, \quad A_{15} = (\hat{E}, \hat{F})^{(2)}, \quad A_{16} = (((\hat{E}, \hat{G})^{(1)}, \hat{C})^{(1)}, \hat{K})^{(2)}, \\ A_{17} &= (((\hat{D}, \hat{D})^{(2)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \quad A_{18} = ((\hat{D}, \hat{F})^{(2)}, \hat{G})^{(1)}, \\ A_{19} &= ((\hat{D}, \hat{D})^{(2)}, \hat{H})^{(2)}, \quad A_{20} = ((\hat{C}, \hat{D})^{(2)}, \hat{F})^{(2)}, \\ A_{21} &= ((\hat{D}, \hat{D})^{(2)}, \hat{K})^{(2)}, \quad A_{22} = (((((\hat{C}, \hat{D})^{(1)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \hat{G})^{(1)}, \\ A_{23} &= ((\hat{F}, \hat{H})^{(1)}, \hat{K})^{(2)}, \quad A_{24} = (((\hat{C}, \hat{D})^{(2)}, \hat{K})^{(1)}, \hat{H})^{(2)}, \\ A_{25} &= ((\hat{D}, \hat{D})^{(2)}, \hat{E})^{(2)}, \quad A_{26} = (\hat{B}, \hat{D})^{(3)}. \end{aligned}$$

Now, we can introduce the following affine invariants

$$\begin{aligned}
C_1 &= 15A_2^2 - 33A_{17} - 8A_{18} - 63A_{19} - 6A_{20} - 9A_{21}, \\
C_2 &= -3A_1A_2 + 2A_{15}, \quad C_3 = A_2, \quad C_4 = A_7, \quad C_5 = -A_2A_3 + 2A_{22}, \\
C_6 &= 12A_1(4A_6 - A_7) + 3A_2(4A_4 - A_3 + 12A_5) + 6A_{22} + 16A_{23}, \\
C_7 &= -20A_1A_7 - A_2A_3 + 2A_{22}, \quad E_1 = A_5, \quad E_2 = A_{25}, \\
C_8 &= -6A_1^2 - 5A_8 - A_{10} - A_{11} - 3A_{12}, \quad C_9 = A_4 - A_5, \\
C_{10} &= A_{26}, \quad C_{11} = A_2^2 - 10A_{17} - 2A_{18} - 6A_{19} + 6A_{21}, \\
C_{12} &= -A_1^2(10A_3 + 9A_5) - 3A_1A_{16} + A_3(30A_8 - 7A_{10} - 5A_{11}) + \\
&\quad + A_4(-22A_8 + 18A_9 - 11A_{10} + 3A_{11}) + 18A_7(3A_6 + 5A_7) + \\
&\quad + 48A_2A_{13} - 2A_6^2 + A_5(46A_8 - 2A_9 + 5A_{10} - 9A_{11}),
\end{aligned} \tag{3}$$

as polynomials in elements $A_1 - A_{26}$.

On the other hand let us consider the following center affine invariants and comitants, which are constructed directly through the right-side parts of system (1):

$$\begin{aligned}
J_1 &= \left| \begin{array}{cc} \partial P_1 / \partial x^1 & \partial P_1 / \partial x^2 \\ \partial Q_1 / \partial x^1 & \partial Q_1 / \partial x^2 \end{array} \right| = a_p^\alpha a_q^\beta \varepsilon_{\alpha\beta} \varepsilon^{pq}, \\
B_1 &= \left| \begin{array}{cc} \partial P_1 / \partial x^1 & \partial P_2 / \partial x^2 \\ \partial Q_1 / \partial x^1 & \partial Q_2 / \partial x^2 \end{array} \right| - \left| \begin{array}{cc} \partial P_1 / \partial x^2 & \partial P_2 / \partial x^1 \\ \partial Q_1 / \partial x^2 & \partial Q_2 / \partial x^1 \end{array} \right| = x^\alpha a_q^\beta a_{p\alpha}^\gamma \varepsilon_{\beta\gamma} \varepsilon^{pq}, \\
B_2 &= \left| \begin{array}{cc} P_0 & P_1 \\ Q_0 & Q_1 \end{array} \right| = x^\alpha a^\beta a_\alpha^\gamma \varepsilon_{\beta\gamma}, \quad B_3 = \frac{1}{4} \left| \begin{array}{cc} \partial P_2 / \partial x^1 & \partial P_2 / \partial x^2 \\ \partial Q_2 / \partial x^1 & \partial Q_2 / \partial x^2 \end{array} \right| = \hat{H}, \\
B_4 &= \left| \begin{array}{cc} P_0 & P_2 \\ Q_0 & Q_2 \end{array} \right| = x^\alpha x^\beta a^\gamma a_{\alpha\beta}^\delta \varepsilon_{\gamma\delta}, \quad B_5 = \left| \begin{array}{cc} P_1 & P_2 \\ Q_1 & Q_2 \end{array} \right| = x^\alpha x^\beta x^\gamma a_\alpha^\delta a_{\beta\gamma}^\mu \varepsilon_{\delta\mu}.
\end{aligned}$$

We shall construct the following transvectants

$$\begin{aligned}
\mu_1 &= (B_3, B_3)^{(2)}, \quad H_1 = (B_3, B_1)^{(1)}, \quad G_1 = (B_1, B_5)^{(1)}, \\
G_2 &= (B_5, B_5)^{(2)}, \quad G_3 = (B_3, B_4)^{(1)}, \quad D_1 = (((\hat{D}, \hat{D})^{(2)}, \hat{D})^{(1)}, \hat{D})^{(3)},
\end{aligned}$$

and introduce the following notations

$$\begin{aligned}
\mu &= -2\mu_1, \quad H = 2H_1, \quad 2G = 4G_1 - 3G_2 + 8G_3, \quad \tilde{S} = B_3, , \\
F &= J_1 B_5 + 2B_1 B_4 + 4B_2 B_3, \quad V = B_4^2 - B_2 B_5, \quad \tilde{N} = \hat{K}, \quad D = -D_1.
\end{aligned} \tag{4}$$

As it was shown in [14], the comitants μ , H , G , F and V are responsible for the number and multiplicities of the finite singular points (FSP) of the quadratic system (1). According to [14] we can construct the following T -comitants:

$$\begin{aligned}
P &= G^2 - 6FH + 12\mu V, \quad R = 4(3H^2 - 2\mu G), \quad S = R^2 - 16\mu^2 P, \\
T &= 2\mu[2G^3 + 9\mu(3F^2 - 8GV) - 18FGH + 108H^2V] - PR, \quad U = F^2 - 4GV.
\end{aligned}$$

We denote by $r_j (c_j)$ any real (complex) FSP of the system (1) of multiplicity j and by m_f [18], we denote the sum of the multiplicities of all FSP (real and complex) of this system.

Proposition 2. [14] The number and multiplicity of the finite singular points of system (1) are determined in Table 1. The sets $M_j \in R^{12}$ ($j = 1, 2, \dots, 19$) which are defined by the conditions given in the third and the fourth columns constitute an affine invariant partition of R^{12} , that is,

$$\bigcup_{j=1}^{19} M_j = R^{12}, \quad M_i \bigcap_{i \neq j} M_j = \emptyset$$

and each set M_j is affine invariant.

Table 1

Singular points		Conditions on		
m_f	multiplicity	invariants	comitants	M_j
4	$r_1 r_1 r_1 r_1$	$\mu \neq 0, D < 0$	$R > 0, S > 0$	M_1
4	$r_1 r_1 c_1 c_1$	$\mu \neq 0, D > 0$	—	M_2
4	$c_1 c_1 c_1 c_1$	$\mu \neq 0, D < 0$	$(R \leq 0) \vee (S \leq 0)$	M_3
4	$r_2 r_1 r_1$	$\mu \neq 0, D = 0$	$T < 0$	M_4
4	$r_2 c_1 c_1$	$\mu \neq 0, D = 0$	$T > 0$	M_5
4	$r_2 r_2$	$\mu \neq 0, D = 0$	$T = 0, PR > 0$	M_6
4	$c_2 c_2$	$\mu \neq 0, D = 0$	$T = 0, PR < 0$	M_7
4	$r_3 r_1$	$\mu \neq 0, D = 0$	$T = 0, P = 0, R \neq 0$	M_8
4	r_4	$\mu \neq 0, D = 0$	$T = 0, P = 0, R = 0$	M_9
3	$r_1 r_1 r_1$	$\mu = 0, D < 0$	$R \neq 0$	M_{10}
3	$r_1 c_1 c_1$	$\mu = 0, D > 0$	$R \neq 0$	M_{11}
3	$r_2 r_1$	$\mu = 0, D = 0$	$R \neq 0, P \neq 0$	M_{12}
3	r_3	$\mu = 0, D = 0$	$R \neq 0, P = 0$	M_{13}
2	$r_1 r_1$	$\mu = 0$	$R = 0, P \neq 0, U > 0$	M_{14}
2	$c_1 c_1$	$\mu = 0$	$R = 0, P \neq 0, U < 0$	M_{15}
2	r_2	$\mu = 0$	$R = 0, P \neq 0, U = 0$	M_{16}
1	r_1	$\mu = 0$	$R = 0, P = 0, U \neq 0$	M_{17}
0	—	$\mu = 0$	$R = P = U = 0, V \neq 0$	M_{18}
∞	—	$\mu = 0$	$R = P = U = 0, V = 0$	M_{19}

Herein we shall use the following assertion from [15] (see Theorem 1.34):

Proposition 3. For the existence of a center of system (1) at the origin of coordinates (i.e. $a^1 = a^2 = 0$) it is necessary and sufficient that the following conditions hold:

$$I_1 = I_6 = 0, \quad I_2 < 0,$$

and that at least one of the following three conditions be met:

$$1) \ I_3 = 0; \ 2) \ I_{13} = 0; \ 3) \ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,$$

where

$$\begin{aligned} I_1 &= a_\alpha^\alpha, \quad I_2 = a_\beta^\alpha a_\alpha^\beta, \quad I_3 = a_p^\alpha a_{\alpha q}^\beta a_{\beta \gamma}^\gamma \varepsilon^{pq}, \quad I_4 = a_p^\alpha a_{\beta q}^\beta a_{\alpha \gamma}^\gamma \varepsilon^{pq}, \\ I_5 &= a_p^\alpha a_{\gamma q}^\beta a_{\alpha \beta}^\gamma \varepsilon^{pq}, \quad I_6 = a_p^\alpha a_{\gamma q}^\beta a_{\alpha q}^\gamma a_{\beta \delta}^\delta \varepsilon^{pq}, \quad I_{13} = a_p^\alpha a_{qr}^\beta a_{\gamma s}^\gamma a_{\alpha \beta}^\delta a_{\delta \mu}^\mu \varepsilon^{pq} \varepsilon^{rs}. \end{aligned}$$

Herein we have used the notations for invariants from [15].

MAIN RESULTS

First we prove the following lemma:

Lemma 1. *For the existence of a center arbitrarily situated in the phase plane of system (1) it is necessary that conditions $C_1 = C_3 = 0$ be satisfied.*

Indeed, let system (1) have a singular point of the center type. After the translation the origin of coordinates to this point we obtain the system

$$\begin{aligned}\frac{dx}{dt} &= cx + dy + ex^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy + lx^2 + 2mxy + ny^2,\end{aligned}$$

for which $C_1 = (c+f)\bar{C}_1$, where \bar{C}_1 is a polynomial on the parameters of this system. Since (according to Proposition 3) the necessary condition for the existence of a center in $(0,0)$ is $I_1 = c+f = 0$ we obtain $C_1 = 0$. Hereby, for the given system we obtain $C_3 = I_6$, and, according to Proposition 3, a center can occur only if $I_6 = 0$, i.e. $C_3 = 0$.

§1. System with total multiplicity $m_f = 4$

Herein we shall find out conditions for the existence of a center by using invariants (3) and Table 1 in the case where the multiplicity m_f equals four. From Table 1 and [19] it follows that the system (1) with $m_f = 4$ may have a center only if it belongs to the set $M_1 \cup M_2 \cup M_4 \cup M_8$. This implies 4 different cases which will be examined in the sequel.

Theorem 1. *System (1) with conditions $\mu \neq 0$, $D < 0$, $R > 0$, $S > 0$ (there are 4 simple singular points) has one center if and only if one of the following two sequences of conditions holds:*

- (i) $C_2C_4 < 0$, $C_1 = C_3 = C_5 = 0$;
- (ii) $C_4 = 0$, $C_1 = C_3 = 0$, $\mu < 0$;

and it has two centers if and only if the following conditions hold

- (iii) $C_4 = 0$, $C_1 = C_3 = 0$, $C_9 \geq 0$, $\mu > 0$.

Proof. Let us assume that system (1) has four real distinct singular points. By applying an affine transformation we can assume that three singular such points are the points $M_0(0,0)$, $M_1(1,0)$ and $M_2(0,1)$, respectively. Hence, system (1) becomes

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2hxy - dy^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy - fy^2,\end{aligned}\tag{5}$$

from which, by using the notations $\bar{C} = cm - eh$, $\bar{D} = de - cf$ and $\bar{F} = fh - dm$, we obtain

$$\mu = \bar{D}^2 - 4\bar{C}\bar{F}, \quad D = -\bar{D}^2(\bar{D} - 2\bar{C})^2(\bar{D} - 2\bar{F})^2.$$

It is easy to observe, that system (5), besides the critical points $M_0(0, 0)$, $M_1(1, 0)$ and $M_2(0, 1)$, has the critical point $M_3(x_0, y_0)$ with coordinates

$$x_0 = \bar{D}(\bar{D} - 2\bar{F})/\mu, \quad y_0 = \bar{D}(\bar{D} - 2\bar{C})/\mu.$$

Following [19], we shall calculate the coefficients of the equation

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

which determine the eigenvalues corresponding to a critical point. Thus, for the critical points M_0 , M_1 , M_2 and M_3 we obtain

$$\begin{aligned} \sigma^{(0)} &= c + f, \quad \Delta^{(0)} = -\bar{D}, \\ \sigma^{(1)} &= -c + f + 2m, \quad \Delta^{(1)} = \bar{D} - 2\bar{C}, \\ \sigma^{(2)} &= c - f + 2h, \quad \Delta^{(2)} = \bar{D} - 2\bar{F}, \\ \sigma^{(3)} &= c + f + 2(m - c)x_0 + 2(h - f)y_0, \\ \Delta^{(3)} &= -\bar{D}(\bar{D} - 2\bar{C})(\bar{D} - 2\bar{F})/\mu, \end{aligned} \tag{6}$$

respectively.

For system (5) we obtain

$$C_1 = 3\mu\sigma^{(0)}\sigma^{(1)}\sigma^{(2)}\sigma^{(4)}$$

and, according to Lemma 1, if $C_1 \neq 0$ system (5) has not a center.

If $C_1 = 0$, in virtue of $\mu \neq 0$ it follows that $\sigma^{(0)}\sigma^{(1)}\sigma^{(2)}\sigma^{(3)} = 0$. Without loss of generality we can suppose that $\sigma^{(0)} = 0$, otherwise we can use the linear transformation, which replaces the points M_0 and M_i in case $\sigma^{(i)} = 0$ ($i = 1, 2, 3$). In all three cases, after the corresponding change of parameters we obtain the same system, but with $\sigma^{(0)} = 0$.

Thus, from $\sigma^{(0)} = 0$ and (6) we get $f = -c$ and for system (5) we have

$$C_4 = \mu\sigma^{(1)}\sigma^{(2)}\sigma^{(3)}/\bar{D}. \tag{7}$$

For the system (5) with the condition $f = -c$, the following affine invariants can be calculated:

$$\begin{aligned} C_4 &= \frac{1}{3}I_{13}^{(0)}, \quad C_2 = 3I_2^{(0)}C_4, \quad C_3 = -\frac{2}{3}I_6^{(0)}, \\ C_5 &= 6I_3^{(0)}C_4, \quad C_6 = 6(5I_3^{(0)} - 2I_4^{(0)})C_4, \quad C_7 = 2(13I_3^{(0)} - 10I_5^{(0)})C_4 \end{aligned} \tag{8}$$

where $I_j^{(0)}$ ($j = 2, 3, 4, 5, 6, 13$) are the values of the center affine invariants from Proposition 3, calculated for system (2) with the singular point $M_0(0, 0)$.

I. If $C_4 \neq 0$, in accordance with (7) we obtain $\sigma^{(1)}\sigma^{(2)}\sigma^{(3)} \neq 0$ and neither of the singular points M_i ($i = 1, 2, 3$) can be a center. In this case, taking into account Proposition 3, the relation $I_1 = c + f = 0$ and (8), we conclude, that the singular point $M_0(0, 0)$ will be a center if and only if the following conditions hold:

$$C_1 = C_3 = C_5(C_6^2 + C_7^2) = 0, \quad C_2C_4 < 0.$$

However, we shall prove that in the case under consideration conditions $C_5 \neq 0$ and $C_6 = C_7 = 0$ can not be satisfied. Indeed, if we suppose the contrary, then from (8) it follows: $I_{13} \neq 0$, $I_2 < 0$, $I_1 = I_6 = 0$, $I_3 \neq 0$ and $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$.

Hereby, as it was shown in [20, p. 131], by applying a center affine transformation system (1) can be brought to the canonical system

$$\frac{dx}{dt} = y + qx^2 + xy, \quad \frac{dy}{dt} = -x - x^2 + 3qxy + 2y^2,$$

for which $D = 8q^2(q^2 + 1) > 0$ in virtue of $I_{13} = 125q(q^2 + 1)/8 \neq 0$. As we can observe this contradicts condition $D < 0$ of Theorem 1.

II. Let condition $C_4 = 0$ be satisfied. From (7) we obtain that $\sigma^{(1)}\sigma^{(2)}\sigma^{(3)} = 0$ and without loss of generality, we can assume that $\sigma^{(1)} = 0$, otherwise we can use a linear transformation. Indeed, if $\sigma^{(2)} = 0$ (resp. $\sigma^{(3)} = 0$) the transformation $x_1 = y$, $y_1 = x$ ($x_1 = x/x_0$, $y_1 = y - xy_0/x_0$) replaces the points M_1 and M_2 (resp. M_1 and M_3) and keeps the other points.

Thus, the conditions $C_1 = C_4 = 0$ imply that $\sigma^{(0)} = \sigma^{(1)} = 0$ and from (6) we receive

$$f = -c, \quad m = c, \quad \sigma^{(2)} = 2(c + h), \quad \sigma^{(3)} = 2(c + h)y_0 \quad (9)$$

and, hence, system (5) becomes as

$$\begin{aligned} \frac{dx}{dt} &= cx + dy - cx^2 + 2hxy - dy^2, \\ \frac{dy}{dt} &= ex - cy - ex^2 + 2cxy + cy^2. \end{aligned} \quad (10)$$

For system (10) the following comitants can be calculated:

$$\begin{aligned} I_1^{(0)} &= I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2(c^2 + de), \\ I_6^{(0)} &= (c + h)(2ceh - 2c^3 - c^2e - de^2). \end{aligned}$$

On the other hand by translating the origin of coordinates at the singular point $M_1(1, 0)$ we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= cx + dy - cx^2 + 2hxy - dy^2, \\ \frac{dy}{dt} &= ex - cy - ex^2 + 2cxy + cy^2, \end{aligned} \quad (11)$$

for which

$$\begin{aligned} I_1^{(1)} &= I_{13}^{(1)} = 0, \quad I_2^{(1)} = 2(c^2 - de - 2eh), \\ I_6^{(1)} &= (c + h)(2ceh - 2c^3 - c^2e - de^2). \end{aligned}$$

For system (10), as well as for system (11) we obtain

$$\begin{aligned} C_3 &= -\frac{2}{3}I_6^{(0)} = -\frac{2}{3}I_6^{(1)}, \\ C_8 &= -\frac{4}{3}(c + h)^2(c^2 + de)(c^2 - de - 2eh) = -\frac{1}{3}(c + h)^2I_2^{(0)}I_2^{(1)}, \\ C_9 &= -(c + h)^2(c^2 - eh) = -\frac{1}{2}(c + h)^2(I_2^{(0)} + I_2^{(1)}), \\ D &= -\frac{1}{18}(c^2 + de)^2(c^2 - de - 2eh)^2(c^2 + de + 2cd + 2ch)^2 \neq 0. \end{aligned} \quad (12)$$

1) If $C_8 \neq 0$, from (12) we obtain $(c + h) \neq 0$ and from (9) we get $\sigma^{(2)}\sigma^{(3)} \neq 0$. Therefore, from (12) and Proposition 3, we conclude, that for $C_8 \neq 0$ system (1) has one center if and only if $C_3 = 0$ and $C_8 > 0$ and has two center if and only if $C_3 = 0$, $C_8 < 0$ and $C_9 > 0$.

It is not too difficult to show that conditions $C_3 = 0$ and $C_8 \neq 0$ imply $\mu C_8 < 0$. Indeed, by virtue of (12) from $C_3 = 0$ and $C_8 \neq 0$ results $e \neq 0$. Therefore we can consider $e = 1$ and, hence, by (12) condition $C_3 = 0$ yields $d = 2ch - 2c^3 - c^2$. Hereby for system (11) we have

$$\mu = -4c(c+1)(c^2-h)^2, \quad C_8 = \frac{16}{3}c(c+1)(c+h)^2(c^2-h)^2,$$

and this implies $\mu C_8 < 0$.

2) Let us assume that the condition $C_8 = 0$ holds. Since condition $D \neq 0$ is satisfied, from (12) we get $(c + h) = 0$, and from (9) and (6) we obtain

$$\sigma^{(0)} = \sigma^{(1)} = \sigma^{(2)} = \sigma^{(3)} = 0, \quad Sgn(\Delta^{(0)}\Delta^{(1)}\Delta^{(2)}\Delta^{(3)}) = Sgn\mu. \quad (13)$$

On the other hand, following Proposition 3 we calculate the values of the invariants I_1 , I_2 , I_6 and I_{13} for the system (5) and for other three systems, that are obtained from (5) by placing the critical points M_1 , M_2 and M_3 at the origin, respectively. Thus, we get the following expressions:

$$\begin{aligned} I_1^{(i)} &= I_6^{(i)} = I_{13}^{(i)} = 0 \quad (i = 0, 1, 2, 3), \\ I_2^{(0)} &= -2\Delta^{(0)}, \quad I_2^{(1)} = -2\Delta^{(1)}, \\ I_2^{(2)} &= -2\Delta^{(2)}, \quad I_2^{(3)} = -2\Delta^{(3)}. \end{aligned} \quad (14)$$

According to Proposition 3, the number of the negative quantities among the $I_2^{(i)}$ ($i = 0, 1, 2, 3$) coincide with the number of centers of system (5).

As it is well known for quadratic system at least one of the quantities $\Delta^{(i)}$ ($i = 0, 1, 2, 3$) is negative and at least one of them is positive. On the other hand, at most two singular points can be of the center or of the focus type. Therefore, from Proposition 3, (13) and (14) we conclude, that system (5) has two centers if $\mu > 0$ and one center if $\mu < 0$. It remain to notice, that condition $C_8 = 0$ (i.e. $c + h = 0$) is equivalent to $C_9 = 0$. Indeed, if we suppose that $c + h \neq 0$, then by (12) condition $C_9 = 0$ yields $c^2 = eh$ and from $C_3 = 0$ it results $e(c^2 + de) = 0$, contrary to the condition $D \neq 0$.

Theorem 1 is proved.

Theorem 2. System (1) with conditions $\mu \neq 0$, $D > 0$ (there are 2 simple real and two imaginary singular points) has one center if and only if one of the following two sequences of conditions holds:

- (i) $C_2C_4 < 0$, $C_1 = C_3 = C_5(C_6^2 + C_7^2) = 0$;
- (ii) $C_4 = 0$, $C_{12} \leq 0$, $C_1 = C_3 = 0$, $\mu > 0$;

and it has two centers if and only if the following sequence of conditions holds

- (iii) $C_4 = 0$, $C_{12} < 0$, $C_1 = C_3 = 0$, $\mu < 0$, $C_9 > 0$.

Proof. Let us assume that system (1) has two real and two imaginary singular points. By applying the affine transformation we can move two real singular points to the points $M_0(0, 0)$ and $M_1(1, 0)$, respectively. Hence, system (1) becomes

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy + ny^2,\end{aligned}\tag{14}$$

for which, by using the notations

$$\begin{aligned}\bar{B} &= cn - ek, \quad \bar{C} = cm - eh \quad \bar{D} = de - cf, \\ \bar{E} &= dn - fk, \quad \bar{F} = fh - dm, \quad \bar{H} = hn - km\end{aligned}$$

we obtain

$$\begin{aligned}\mu &= \bar{B}^2 + 4\bar{C}\bar{H} \neq 0, \quad D = -\frac{2}{27}\bar{D}^2(\bar{D} - 2\bar{C})^2Z > 0, \\ Z &= (\bar{B} - 2\bar{F})^2 + 4\bar{E}(\bar{D} - 2\bar{C}).\end{aligned}\tag{15}$$

Since $D > 0$, it follows that $Z < 0$ and for the real singular points $M_0(0, 0)$ and $M_1(1, 0)$ we obtain

$$\sigma^{(0)} = c + f, \quad \sigma^{(1)} = -c + f + 2m.\tag{16}$$

For system (14) we can calculate that

$$\begin{aligned}\mu C_1 &= 3\sigma^{(0)}\sigma^{(1)}(R^2 - ZI^2), \quad \mu C_4 = \bar{P}R + \bar{Q}I, \\ C_{12} &= \frac{1}{4}[\mu^2(\sigma^{(0)} - \sigma^{(1)})^2 - 4mR(\sigma^{(0)} + \sigma^{(1)}) + 4ZI^2],\end{aligned}\tag{17}$$

where

$$\begin{aligned}R &= (c - m)(2\bar{B}\bar{F} + 4\bar{C}\bar{E} - \bar{B}^2) - 2(h + n)(\bar{B}\bar{C} - \bar{B}\bar{D} + 2\bar{C}\bar{F}) + \\ &\quad + (c + f)\mu, \quad I = (m - c)\bar{B} - 2(h + n)\bar{C}, \\ \bar{P} &= \bar{B}(-cn - hm + mn) - \bar{C}[(h - n)^2 + k(c - 2m)] + \bar{H}m^2, \\ \bar{Q} &= \bar{B}^2(2fh - cd - ch + 2hm) + \bar{B}\bar{D}(-ck - h^2 + 2hn - n^2) + \\ &\quad + \bar{B}\bar{C}(ck - 4dh - 2fk - 3h^2 - 2hn - 2km + n^2) - \bar{B}\bar{F}m(2h + n) - \\ &\quad - \bar{B}\bar{H}m(f + m) + 2\bar{C}^2k(d + 2h + 2n) - 2\bar{C}\bar{D}k(h + 2n) + \\ &\quad + 2\bar{C}\bar{F}(h + n)^2 + 4\bar{C}\bar{H}m(d + h + n) - 4\bar{D}\bar{H}mn + 2\bar{F}\bar{H}m^2.\end{aligned}\tag{18}$$

According to Lemma 2, if $C_1 \neq 0$ system (14) has not a center.

Let us assume that condition $C_1 = 0$ holds.

Case I. If $C_4 \neq 0$ from (18) we get $R^2 + I^2 \neq 0$ and by $Z < 0$ and (17) the condition $C_1 = 0$ implies that $\sigma^{(0)}\sigma^{(1)} = 0$. Without loss of generality we can assume that $\sigma^{(0)} = 0$, otherwise we can use the linear transformation, which replaces the points M_0 and M_1 .

Thus, $\sigma^{(0)} = 0$, and from (16), it follows that $f = -c$ and for system (14) we obtain

$$12\mu\bar{D}C_4 = \sigma^{(1)}(R^2 - ZI^2).\tag{19}$$

On the other hand, for system (14) with the condition $f = -c$, the following affine invariants can be calculated:

$$\begin{aligned} C_4 &= \frac{1}{3}I_{13}^{(0)}, \quad C_2 = 3I_2^{(0)}C_4, \quad C_3 = -\frac{2}{3}I_6^{(0)}, \\ C_5 &= 6I_3^{(0)}C_4, \quad C_6 = 6(5I_3^{(0)} - 2I_4^{(0)})C_4, \quad C_7 = 2(13I_3^{(0)} - 10I_5^{(0)})C_4, \end{aligned} \quad (20)$$

where $I_j^{(0)}$ ($j = 2, 3, 4, 5, 6, 13$) are the values of the center affine invariants from Proposition 3, calculated for system (14) with singular point $M_0(0, 0)$.

Since $\mu D \neq 0$, by (19) condition $C_4 \neq 0$ implies that $\sigma^{(1)} \neq 0$, i.e. singular point M_1 is not a center.

Thus, in the case of $C_4 \neq 0$, from Proposition 3 and relationship $I_1 = c + f = 0$ and (20) we conclude, that the system (14) has one center if and only if the following conditions hold:

$$C_1 = C_3 = C_5(C_6^2 + C_7^2) = 0, \quad C_2C_4 < 0.$$

Case II. Let condition $C_4 = 0$ be satisfied.

A) If $C_8 \neq 0$ we shall examine two cases: $C_{12} < 0$ and $C_{12} \geq 0$.

1) Let us consider firstly that condition $C_{12} < 0$ holds. Then $R^2 + I^2 \neq 0$, otherwise from (17) we get $C_{12} = \mu^2(\sigma^{(0)} - \sigma^{(1)})^2 \geq 0$. Therefore, $R^2 - ZI^2 \neq 0$ and conditions $C_1 = C_4 = 0$, (17) and (19) imply that $\sigma^{(0)} = \sigma^{(1)} = 0$. From (16) it follows that $f = -c$, $m = c$ and the system (14) becomes as

$$\begin{aligned} \frac{dx}{dt} &= cx + dy - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex - cy - ex^2 + 2cxy + ny^2. \end{aligned} \quad (21)$$

For system (21) the following invariants can be calculated:

$$\begin{aligned} I_1^{(0)} &= I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2(c^2 + de), \\ I_6^{(0)} &= (h + n)(2ceh - 2c^3 - cen + e^2k). \end{aligned}$$

On the other hand, by translating the origin of coordinates to the singular point $M_1(1, 0)$ of system (21) we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= -cx + (d + 2h)y - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= -ex + cy - ex^2 + 2cxy + ny^2, \end{aligned} \quad (22)$$

for which

$$\begin{aligned} I_1^{(1)} &= I_{13}^{(1)} = 0, \quad I_2^{(1)} = 2(c^2 - de - 2eh), \\ I_6^{(1)} &= (h + n)(2ceh - 2c^3 - cen + e^2k). \end{aligned}$$

For systems (21) and (22) we obtain

$$\begin{aligned} C_3 &= -\frac{2}{3}I_6^{(0)} = -\frac{2}{3}I_6^{(1)}, \\ C_8 &= -\frac{4}{3}(h+n)^2(c^2+de)(c^2-de-2eh) = -\frac{1}{3}(h+n)^2I_2^{(0)}I_2^{(1)}, \\ C_9 &= (h+n)^2(eh-c^2) = -\frac{1}{2}(h+n)^2(I_2^{(0)} + I_2^{(1)}), \\ D &= -\frac{2}{27}(c^2+de)^2(c^2-de-2eh)^2Z \neq 0, \\ C_{12} &= I^2Z = 4(h+n)^2(c^2-eh)^2Z. \end{aligned} \quad (23)$$

From $C_{12} \neq 0$ and (23) we get $(h+n) \neq 0$. Therefore, from (23) and Proposition 3, we conclude that for $C_1 = C_4 = 0$ and $C_{12} < 0$ system (1) has one center if and only if $C_3 = 0$ and $C_8 > 0$ and has two centers if and only if $C_3 = 0$, $C_8 < 0$ and $C_9 > 0$. It is not difficult to convince, that in the case under consideration condition $C_8 < 0$ ($C_8 > 0$) is equivalent to $\mu < 0$ ($\mu > 0$). Indeed, condition $C_4 = 0$ by (20) yields $I_{13} = 0$ and it was shown in [20] that by applying a linear transformation system (1) with can be brought either to the canonical system [20, p. 103]

$$\frac{dx}{dt} = y + 2(1-c)xy, \quad \frac{dy}{dt} = -x + dx^2 + cy^2. \quad (24)$$

or to the canonical system [20, p. 80]

$$\frac{dx}{dt} = -y - cx^2 - ay^2, \quad \frac{dy}{dt} = x + bx^2 + 2cxy, \quad (25)$$

For system (24) one can be calculated

$$\begin{aligned} \mu &= 4cd(c-1)^2, \quad D = \frac{1}{3}c(d+2-2c)^3, \quad C_{12} = -2 < 0, \\ C_8 &= 4d(d+2-2c), \quad C_1 = C_3 = 0. \end{aligned}$$

Hereby by virtue of $D > 0$ it results $\mu C_8 > 0$ and since $C_{12} < 0$ this has proved our affirmation.

As regards system (25) we obtain

$$\mu = a^2b^2 + 4ac^3, \quad D = \frac{1}{6}(a-2c)^2(4ac-b^2-8c^2), \quad C_1 = C_3 = C_8 = C_{12} = 0,$$

i.e. condition $C_8 \neq 0$ is not satisfied.

2) Let condition $C_{12} \geq 0$ holds. We shall demonstrate that by virtue $C_8 \neq 0$ there is no center on the phase plane of system (14). Indeed, according to (17), the condition $C_1 = 0$ implies that $\sigma^{(0)}\sigma^{(1)}(R^2 - ZI^2) = 0$.

If $R^2 + I^2 \neq 0$ then from $C_1 = C_4 = 0$, $Z < 0$ and (19) we get $\sigma^{(0)} = \sigma^{(1)} = 0$, i.e. $f = -c$, $m = c$. Therefore, for system (14) we obtain

$$C_{12} = ZI^2, \quad C_3 = \frac{2}{3}(h+n)(2c^3 - 2ceh + cen - e^2k), \quad R = 3(d+h)C_3,$$

and conditions $C_{12} \geq 0$, $Z < 0$ and $R^2 + I^2 \neq 0$ imply $I = 0$, $R \neq 0$. Hereby, we get that $C_3 \neq 0$ and by Lemma 1 system (14) has no singular point of the center type. We note that, according to (23) and $D \neq 0$, in this case $C_8 \neq 0$.

Let conditions $R = I = 0$ be satisfied. Without loss of generability we can assume that relation $ce = 0$ holds. Indeed, if $e \neq 0$, by applying, the transformation $x_1 = x - cy/e$, $y_1 = y$, which keeps the singular points $M_0(0, 0)$ and $M_1(1, 0)$ for system (14) the relation $c = 0$ will be satisfied. Thus, we shall consider two cases: $e \neq 0, c = 0$ and $e = 0$.

a) If $e \neq 0, c = 0$ for system (14) we have $I = e[2h^2 + 2hn - km] = 0$.

$\alpha)$ If $m \neq 0$ by using the change of the time the condition $m = 1$ will be satisfied. Therefore, from $I = 0$, we obtain $k = 2h^2 + 2hn$ and for system (14) in this case one can derive

$$R = 4eh(fh - d)[e(h + n)^2 + 2h + n], \quad \mu = 4eh^2[e(h + n)^2 + 2h + n].$$

Since $R = 0$, $\mu \neq 0$ we get $d = fh$. Thus, for system (14) we obtain

$$\begin{aligned} C_3 &= -\frac{4}{3}he[e(h + n)^2 - f(f + 2)(2h + n)], \quad Z = -3hC_3, \\ C_8 &= \frac{1}{3}f(f + 2)\mu, \quad D = -\frac{2}{27}e^4h^4f^2(f + 2)^2Z, \end{aligned} \tag{26}$$

and by $Z < 0$ it follows $C_3 \neq 0$ and hence, according to Lemma 1 system (14) has not any center. Notice, that in this case from (26) and $D \neq 0$ we get $C_8 \neq 0$.

$\beta)$ If $m = 0$ since $c = 0$ for system (14), we have

$$I = 2eh(h + n) = 0, \quad C_8 = \frac{4}{3}[3fR + de^2(d + 2h)(h + n)^2], \tag{27}$$

and from $I = R = 0$, $C_8 \neq 0$ and (27) it follows $h = 0$, $n \neq 0$. Therefore, for system (14) with $c = m = h = 0, ne \neq 0$ we obtain:

$$R = e^2k(fk - 2dn) = 0, \quad \mu = e^2k^2 \neq 0.$$

Thus, the condition $k \neq 0$ is satisfied and we can consider $k = 1$ by a change of scale if necessary. Condition $R = 0$ implies $f = 2dn$ and for system (14) the following polynomials can be calculated:

$$C_3 = \frac{2}{3}en(4d^2n - e), \quad Z = -e(4d^2n - e).$$

Hereby, from $Z < 0$ and $ne \neq 0$, it follows that $C_3 \neq 0$ and from Lemma 1 we can conclude that there is not any center on the phase plane of system (14) in this case under consideration.

b) If $e = 0$ for system (14) we have

$$\mu = c(cn^2 + 4hmn - 4km^2), \quad I = -c(cn + 2hm + mn),$$

hence, the conditions $I = 0$, $\mu \neq 0$ imply $m \neq 0$, $cn + 2hm + mn = 0$. As it was mentioned above, we can assume $m = 1$ hence, $h = -\frac{1}{2}n(c + 1)$. Therefore, for system (14) we obtain

$$\mu = -c(cn^2 + 4k + 2n^2), \quad R = c(f + 1)\mu,$$

and from conditions $\mu \neq 0$, $R = 0$ we get $f = -1$. Hereby for system (14) the following relation can be established

$$\sigma^{(0)} = -\sigma^{(1)} = c - 1, \quad C_8 = -\frac{1}{3}(c - 1)^2\mu \neq 0. \quad (28)$$

From $C_8 \neq 0$ it follows that $\sigma^{(0)}\sigma^{(1)} \neq 0$ and, hence, there is no singular point of the center type for system (14).

B) Let us now assume that the condition $C_8 = 0$ is satisfied. As it was indicated above from $C_1 = C_4 = 0$ and $R^2 + I^2 \neq 0$ it follows that $C_8 \neq 0$. Hence, for system (14) we obtain $R = I = 0$ and we again shall examine two cases: $e \neq 0, c = 0$ and $e = 0$.

a) Let the conditions $e \neq 0, c = 0$ hold. If $m \neq 0$ it was shown above that from (26) and $D\mu \neq 0$ it follows that $C_8 \neq 0$. Hence, $m = 0$, and taking into account (27) and the relations $R = I = 0, C_8 = 0$ for system (14) we obtain:

$$I = 2eh(h + n) = 0, \quad C_8 = \frac{4}{3}de^2(d + 2h)(h + n)^2, \quad \mu = e(ek^2 - 4h^2n).$$

From $I = 0, C_8 = 0$ we get $h = -n \neq 0$. Indeed, if we suppose that condition $h + n \neq 0$ is satisfied, then from $I = C_8 = 0$ it follows that $h = 0, n \neq 0, de = 0$ and we obtain a contradiction with condition $D = -\frac{2}{27}d^4e_4Z \neq 0$. Thus, $h + n = 0$ and for the system (14) with $c = m = 0, n = -h \neq 0$ one can calculate:

$$\mu = e(ek^2 + 4h^3) \neq 0, \quad C_8 = 2f^2\mu.$$

Hereby, conditions $C_8 = 0, \mu \neq 0$ imply $f = 0$ and the system (14) becomes as system

$$\frac{dx}{dt} = dy + 2hxy + ky^2, \quad \frac{dy}{dt} = ex - ex^2 - hy^2, \quad (29)$$

for which

$$C_1 = C_3 = C_{12} = 0, \quad Z = e(ek^2 - 4d^2h - 8dh^2) < 0, \quad \mu = e(ek^2 + 4h^3) \neq 0. \quad (30)$$

For system (29) the following invariants (from Proposition 3) can be calculated.

$$I_1^{(0)} = I_6^{(0)} = I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2de.$$

On the other hand by translating the origin of coordinates at the singular point $M_1(1, 0)$ of system (29) we obtain the system

$$\frac{dx}{dt} = (d + 2h)y + 2hxy + ky^2, \quad \frac{dy}{dt} = -ex - ex^2 - hy^2, \quad (31)$$

for which

$$I_1^{(1)} = I_6^{(1)} = I_{13}^{(1)} = 0, \quad I_2^{(1)} = -2(de + 2eh).$$

It is easy to observe, that

$$I_2^{(0)}I_2^{(1)} = -4de^2(d + 2h), \quad I_2^{(0)} + I_2^{(1)} = -4eh, \quad Sgn\mu = Sgn(eh). \quad (32)$$

Indeed, from (30) it follows that

$$\mu - Z = 4eh(d + h)^2,$$

and since $Z < 0$ the condition $\mu > 0$ implies $eh > 0$. On the other hand from $\mu = e^2k^2 + 4eh^3 < 0$ it results $eh < 0$. Therefore, taking into account condition $Z = e^2k^2 - 4deh(d + 2h) < 0$ we obtain that $deh(d + 2h) > 0$ and, hence, from (32) it follows that

$$Sgn[\mu I_2^{(0)} I_2^{(1)}] = -Sgn[de^3h(d + 2h)] = -1, \quad Sgn\mu = -Sgn(I_2^{(0)} + I_2^{(1)}).$$

Hereby, we have obtained, that for $\mu > 0$ it results $I_2^{(0)} I_2^{(1)} < 0$ and, according to Lemma 1, either singular point M_0 or M_1 is of the center type. If $\mu < 0$ then $I_2^{(0)} I_2^{(1)} > 0$ and $I_2^{(0)} + I_2^{(1)} > 0$. Thus, both quantities $I_2^{(0)}$ and $I_2^{(1)}$ are positive and by Lemma 1 there is no center on the phase plane of system (29).

b) Let us assume that condition $e = 0$ hold. As it was mentioned above, by virtue of conditions $\mu \neq 0$, $I = R = 0$ we get that $m = 1$, $h = -\frac{1}{2}n(c + 1)$ and $f = -1$. Hereby, we obtain (28) for system (14) and condition $C_8 = 0$ in this case implies that $c = 1$. Therefore, we obtain the system

$$\begin{aligned} \frac{dx}{dt} &= x + dy - x^2 + 4xy + ky^2, \\ \frac{dy}{dt} &= -y + 2xy - 2y^2, \end{aligned} \tag{33}$$

for which

$$\mu - Z = -4(d - 2n)^2 \leq 0, \quad C_{12} = 0.$$

Hence, $\mu < 0$ because $Z < 0$. Thus, for system (33) as well as for the translated system (with singular point M_1 at the origin of coordinates) we obtain that $I_2^{(0)} = I_2^{(1)} = 2 > 0$. By Proposition 3 system (33) has no a center.

Thus we have found out, that in the case $C_1 = C_3 = C_4 = C_8 = 0$ system (14) have a center if and only if the condition $\mu > 0$ hold. Moreover, the center is unique. It remains to note, that for $C_1 = C_3 = C_4 = 0$ and $\mu D \neq 0$ conditions $C_8 = 0$ and $C_{12} = 0$ are equivalent, since for system (24) we have $C_8 C_{12} \neq 0$ and for system (25) the conditions $C_8 = C_{12} = 0$ hold.

As all cases were examined, Theorem 2 is proved.

Theorem 3. *For the existence of a center of system (1) with $\mu \neq 0$, $D = 0$, $T < 0$ (there are two simple and one double singular points) it is necessary and sufficient that one of the following two sequences of conditions holds:*

- (i) $C_2 C_4 < 0$, $C_1 = C_3 = C_5 = 0$;
- (ii) $C_4 = 0$, $\mu > 0$, $C_1 = C_3 = C_8 = 0$.

Moreover, the center is unique.

Proof. According to Proposition 2 if conditions $\mu \neq 0$, $D = 0$, $T < 0$ are satisfied the system (1) has two simple and one double singular points situated on its phase plane. We shall find out the canonical form of a system (1) with such

points. By applying the affine transformation we can move these three singular points to the points $M_0(0,0)$, $M_1(1,0)$ and $M_2(0,1)$, respectively. Hence, system (1) can be brought to the system

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2hxy - dy^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy - fy^2,\end{aligned}\tag{34}$$

for which, by using the notations $\hat{C} = cm - eh$, $\hat{D} = de - cf$ and $\hat{F} = fh - dm$, we have

$$\mu = \hat{D}^2 - 4\hat{C}\hat{F}, \quad D = -\hat{D}^2(\hat{D} - 2\hat{C})^2(\hat{D} - 2\hat{F})^2.$$

It is easy to observe, that system (34) has, besides the critical points $M_0(0,0)$, $M_1(1,0)$ and $M_2(0,1)$ the critical point $M_3(x_0, y_0)$, where

$$x_0 = \frac{1}{\mu} \hat{D}(\hat{D} - 2\hat{F}), \quad y_0 = \frac{1}{\mu} \hat{D}(\hat{D} - 2\hat{C}).$$

Thus, we can conclude that in virtue of $D = 0$ the point M_3 will coincide with M_0 for $\hat{D} = 0$, with point M_1 for $\hat{D} = 2\hat{C}$ and with M_2 for $\hat{D} = 2\hat{F}$. Without loss of generality we can assume that the condition $D = 0$ implies that $\hat{D} = de - cf = 0$ and that the singular point M_0 becomes degenerated (we can removed the respective points if it is necessary). Therefore, without loss of generality one can sets $d = cu$, $f = eu$ and system (34) becomes

$$\begin{aligned}\frac{dx}{dt} &= cx + cuy - cx^2 + 2hxy - cuy^2, \\ \frac{dy}{dt} &= ex + euy - ex^2 + 2mxy - euy^2.\end{aligned}\tag{35}$$

By [19], for the critical points M_1 and M_2 of system (35) we must have

$$\sigma^{(1)} = -c + eu + 2m, \quad \sigma^{(2)} = c - eu + 2h,\tag{36}$$

respectively.

For system (35) we obtain

$$\mu = 4u(cm - eh)^2 \neq 0, \quad C_1 = 3\mu(c + eu)^2\sigma^{(1)}\sigma^{(2)}.\tag{37}$$

If $C_1^2 + C_3^2 \neq 0$ Lemma 1 implies that system (35) has no center.

Let us assume $C_1 = C_3 = 0$ and examine two cases: $C_4 \neq 0$ and $C_4 = 0$.

1) If $C_4 \neq 0$ then the condition $c + eu \neq 0$ holds. Indeed, if $c = -eu$ for system (35) we obtain that

$$\begin{aligned}\mu &= 4e^2u(h + mu)^2 \neq 0, \quad C_1 = 0, \quad C_3 = \frac{4}{3}e^2u(h + mu)(h - eu^2 - eu - mu), \\ C_4 &= \frac{2}{3}e(h + mu)(h - eu^2 - eu - mu)(ue + m)(h - ue),\end{aligned}\tag{38}$$

and from $\mu \neq 0$ and $C_3 = 0$ we obtain that $C_4 = 0$.

Thus, $c + eu \neq 0$ and from (37) the condition $C_1 = 0$ implies $\sigma^{(1)}\sigma^{(2)} = 0$. Without loss of generality, we can assume that $\sigma^{(1)} = 0$, otherwise, if $\sigma^{(2)} = 0$ we can use the linear transformation $x_1 = y$, $y_1 = x$, which replaces the points M_1 and M_2 and keeps the canonical form of system (35). Thus, after the respective changing of parameters, we derive the same system, but with $\sigma^{(1)} = 0$. In this case from (37) we obtain $c = eu + 2m$ and for the system (35) we have

$$\begin{aligned}\mu &= 4u(eh - eum - 2m^2)^2, \\ C_3 &= \frac{4}{3}u(eh - eum - 2m^2)(eh + 3em + 6emu + 2e^2u^2 + 2e^2u + 4m^2).\end{aligned}$$

Since $\mu \neq 0$ from $C_3 = 0$ it results

$$e(h + 3m + 6mu + 2eu^2 + 2eu) + 4m^2 = 0.$$

Therefore, we have $e \neq 0$, otherwise it follows $m = 0$ and, hence, $\mu = 0$. Thus, we can assume that $e = 1$ (by scaling time if necessary) and we get the relation: $h = -3m - 6mu - 2u^2 - 2u - 4m^2$. In this case after a shift of the origin of coordinates to the singular point $M_1(1, 0)$ of system (35) we get the system

$$\begin{aligned}\frac{dx}{dt} &= -(u + 2m)x + [(u + 2m)u + 2\bar{h}]y - (u + 2m)x^2 + 2\bar{h}xy - (u + 2m)uy^2, \\ \frac{dy}{dt} &= -x + (u + 2m)y - x^2 + 2mxy - uy^2\end{aligned}\tag{39}$$

where $\bar{h} = -3m - 6mu - 2u^2 - 2u - 4m^2$. From Proposition 3 applied to the system (39) we obtain that

$$\begin{aligned}I_1 &= I_6 = 0, \quad I_2 = 4(2u + 3m)(u + 2m + 1), \\ I_3 &= (u + m)(u + 2m)(u + 2m + 1)(5u + 12m + 9), \\ I_{13} &= -4u(u + m)^3(2u + 3m)(u + 2m + 1)^2, \\ 5I_3 - 2I_4 &= (u + m)(u + 2m + 1)(5u + 12m + 9)(7u + 12m), \\ 13I_3 - 10I_5 &= -(u + 2m + 1)(48m^3 + 32m^2u + 36m^2 + 9mu^2 \\ &\quad + 9mu + 5u^3 - 7u^2).\end{aligned}\tag{40}$$

On the other hand the following affine invariants can be calculated for system (39)

$$\begin{aligned}\mu &= 4u(2u + 3m)^2(u + 2m + 1)^2, \\ C_4 &= \frac{1}{3}I_{13}, \quad C_2 = 3I_2C_4, \quad C_5 = 6I_3C_4.\end{aligned}\tag{41}$$

It is easy to show, from (40), (41) and $C_4 \neq 0$ that conditions $I_2 < 0$ and $I_3 = 0$ are equivalent to $C_2C_4 < 0$ and $C_5 = 0$, respectively. Note that conditions $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$ can not be satisfied for system (39) because of $C_4 \neq 0$. Indeed, from $5I_3 - 2I_4 = 0$ and (40) it follows $(5u + 12m + 9)(7u + 12m) = 0$, however, from (40) it is easy to observe that in both determined by two factors cases we have $13I_3 - 10I_5 \neq 0$.

Notice also that given the indicated values of parameters c, e and h from $C_4 \neq 0$, (41) and (36) it follows that $\sigma^{(2)} = -4(m+u)(2m+u+1) \neq 0$. Hence, by [19] and Lemma 1 there exists only one singular point of the center type for system (35).

Thus, in the case $C_4 \neq 0$ the assertion of Theorem 3 is valid.

2) Let us assume now $C_4 = 0$. In this case the condition $c + eu = 0$ holds, otherwise, as it was demonstrated above, from $C_1 = C_3 = 0$ we obtain that $c = eu + 2m$, $e = 1$, $h = -3m - 6mu - 2u^2 - 2u - 4m^2$, and

$$\mu = 4u(2u + 3m)^2(u + 2m + 1)^2, \quad C_4 = -\frac{4}{3}u(u + m)^3(2u + 3m)(u + 2m + 1)^2.$$

Hereby in virtue of $\mu \neq 0$ and $c + eu = 2(m + u) \neq 0$ it follows that $C_4 \neq 0$. This contradiction proves our assertion.

Thus, condition $c = -eu$ is satisfied and hence, we arrive to the system

$$\begin{aligned} \frac{dx}{dt} &= -eux - eu^2y + eux^2 + 2hxy + eu^2y^2, \\ \frac{dy}{dt} &= ex + euy - ex^2 + 2mxy - euy^2, \end{aligned} \tag{42}$$

for which

$$\mu = 4e^2u(h + mu)^2 \neq 0, \quad C_1 = 0, \quad C_3 = 4e^2u(h + mu)(h - eu^2 - eu - mu).$$

Therefore, from condition $C_3 = 0$ and $\mu \neq 0$, we obtain $h = eu^2 + eu + mu$. In this case, for system (42) we also obtain

$$\begin{aligned} \sigma^{(1)} &= 2(eu + m), \quad \mu = 4e^2u^3(eu + e + 2m)^2, \\ \sigma^{(2)} &= 2u(eu + m), \quad C_8 = \frac{16}{3}e^2u^4(eu + m)^2(eu + e + 2m)^2. \end{aligned} \tag{43}$$

According to [19] for the existence of a center for system (42) it is necessary that $m = -eu$ and this condition together with (43) and $\mu \neq 0$, is equivalent to $C_8 = 0$. In this case $\mu = 4e^4u^3(u - 1)^2$, and after shift of the origin of coordinates to the singular point $M_1(1, 0)$ system (42) can be transformed into the system

$$\begin{aligned} \frac{dx}{dt} &= eux + eu(2 - u)y + eux^2 + 2euxy + eu^2y^2, \\ \frac{dy}{dt} &= -ex - euy - ex^2 - 2euxy - euy^2, \end{aligned}$$

for which

$$I_1^{(1)} = I_6^{(1)} = I_{13}^{(1)} = 0, \quad I_2^{(1)} = 4e^2u(u - 1).$$

On the other hand, after a shift of the origin of coordinates to the singular point $M_2(0, 1)$ in the case under consideration system (42) will be brought to the system

$$\begin{aligned} \frac{dx}{dt} &= eux + eu^2y + eux^2 + euxy + eu^2y^2, \\ \frac{dy}{dt} &= e(1 - 2u)x - euy - ex^2 - 2euxy - euy^2, \end{aligned}$$

for which

$$I_1^{(2)} = I_6^{(2)} = I_{13}^{(2)} = 0 \quad I_2^{(2)} = 4e^2u^2(1-u).$$

As it is easily seen $I_2^{(1)}I_2^{(2)} = -4\mu^2$. Therefore, if $\mu > 0$ we get $I_2^{(1)}I_2^{(2)} < 0$ and by Lemma 1 the system (42) has only one singular point of center type. If $\mu < 0$ we obtain $u < 0$ and it is easy to see that in this case $I_2^{(1)} > 0$, $I_2^{(2)} > 0$. By Lemma 1, system (42) has not any centers.

Theorem 3 is proved.

Theorem 4. *For the existence of a center of system (1) with $\mu \neq 0, D = T = P = 0, R \neq 0$ (there are one simple and one triple singular points) it is necessary and sufficient that the following conditions hold:*

$$\begin{aligned} C_3 &= C_4 = 0, \quad C_9 > 0; \\ C_3 &= C_4 = C_9 = 0, \quad \mu > 0. \end{aligned}$$

Moreover, the center is unique.

Proof. Let us assume that system (1) has one simple and one triple singular points. By applying the affine transformation we can move the real singular points to the points $M_0(0, 0)$ and $M_1(1, 0)$, respectively. Hence, system (1) becomes

$$\begin{aligned} \frac{dx}{dt} &= cx + dy - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy + ny^2, \end{aligned} \tag{44}$$

for which

$$\mu = (cn - ek)^2 + 4(cm - eh)(hn - km) \neq 0.$$

Without loss of generality we can assume, that the critical point $M_0(0, 0)$ is degenerated (of the third multiplicity). Therefore, the condition $cf - de = 0$ holds. For $\mu \neq 0$ it follows that $c^2 + e^2 \neq 0$ and one can sets $c = d = 0$. Indeed, these relations can be obtained by the transformation $x_1 = ex - cy$, $y_1 = y$ if $e \neq 0$ and by the transformation $x_1 = y$, $y_1 = x$ if $e = 0$ (in this case the conditions $cf - de = 0$ and $c \neq 0$ imply $f = 0$).

Thus, for system (44) with $c = d = 0$ the following comitants can be calculated:

$$P = e^2(ek - 2fh)^2(2hx + ky)^2y^2, \quad \mu = e(ek^2 - 4h^2n + 4hkm).$$

From $\mu \neq 0$ we obtain $e \neq 0$ and we can assume $e = 1$ (scaling the time if necessary) and, hence, the condition $P = 0$ implies $k = 2fh$. Therefore, we get the canonical form

$$\frac{dx}{dt} = 2hxy + 2fhy^2, \quad \frac{dy}{dt} = x + fy - x^2 + 2mxy + ny^2, \tag{45}$$

for which

$$\mu = 4h^2(f^2 + 2fm - n), \quad C_3 = 2hf(f^2 + 2fm - n). \tag{46}$$

According to Lemma 1 for the existence of a center it is necessary that $C_3 = 0$. From (46) and condition $\mu \neq 0$ we obtain $f = 0$ and then for this system $C_1 = 0$. Hereby,

after shifting of the origin of coordinates to the simple singular point $M_1(1,0)$ of system (45) we get the following system

$$\frac{dx}{dt} = 2hxy + 2fh^2y^2, \quad \frac{dy}{dt} = x + fy - x^2 + 2mxy + ny^2, \quad (47)$$

for which

$$\begin{aligned} I_1 &= m, \quad I_2 = 4(m^2 - h), \quad \mu = -4h^2n, \\ C_4 &= \frac{2}{3}mh(h+n)^2, \quad C_9 = h[(h+n)^2 + m^2n]. \end{aligned} \quad (48)$$

If $C_4 \neq 0$ we obtain $I_1 \neq 0$ and according to Proposition 3, the singular point $(0,0)$ of system (47) (i.e. point $M_1(1,0)$ of system (45)) is not a center.

Let us assume now that the condition $C_4 = 0$ is satisfied. By (48) and $\mu \neq 0$ it follows that $m(h+n) = 0$.

1) If $C_9 > 0$ the condition $m = 0$ holds, otherwise for $h = -n$ we obtain $C_9 = -m^2n^2 \leq 0$. Therefore, $I_1 = 0$ and $I_2 < 0$ because of $SgnI_2 = -Sgnh = -SgnC_9$.

Notice that for system (48) with $m = 0$ the conditions $I_6 = I_{13} = 0$ are satisfied and, hence, by Proposition 3 system (48) has one center if $I_1 = 0$, $I_2 < 0$, i.e. if $C_4 = 0$, $C_9 > 0$.

2) If $C_9 < 0$ by (48) and $\mu \neq 0$ we obtain that either $m = 0$, $h < 0$ (i.e. $I_1 = 0$, $I_2 > 0$) or $h = -n$, $-m^2n^2 < 0$ (i.e. $I_1 \neq 0$). In both cases, by Proposition 3 system (47) has no center.

2) If $C_9 = 0$ the conditions $m(h+n) = 0$, $(h+n)^2 + m^2n = 0$ and $hn \neq 0$ imply $m = 0$, $n = -h$. Hereby, $\mu = 4h^3$ and $SgnI_2 = -Sgn\mu$. Therefore, if $C_4 = C_9 = 0$, $\mu > 0$ for system (47) we obtain that $I_1 = I_6 = I_{13} = 0$, $I_2 < 0$ and hence, according to Proposition 3, system (48) has a singular point of a center type.

Theorem 4 is proved.

§2. System with total multiplicity $m_f = 3$

In this section we shall determine the conditions for the existence of a center by using the invariants (3) and Table 1 in the case where multiplicity m_f equals three.

From Table 1 and [19] it follows that the system (1) with $m_f = 3$ can have a center only if it belongs to set $M_{10} \cup M_{11} \cup M_{12}$. This implies 3 different cases which will be examined in the sequel.

Theorem 5. *For the existence of a center of system (1) with $\mu = 0$, $H \neq 0$, $D < 0$ (there are three simple singular points) it is necessary and sufficient that one of the following two sequences of conditions holds:*

- (i) $C_2C_4 < 0$, $C_1 = C_3 = C_5 = 0$;
- (ii) $C_4 = 0$, $C_1 = C_3 = C_{10} = 0$, $C_{11} \leq 0$.

Moreover, the center is unique.

Proof. Let us assume that system (1) has three simple singular points. By applying an affine transformation we can replace two real singular points by the

points $M_0(0,0)$ and $M_1(1,0)$, respectively. Hence, system (1) will be transformed into the system

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy + ny^2,\end{aligned}\tag{49}$$

for which, by using the notations

$$\begin{aligned}\bar{B} &= cn - ek, \quad \bar{C} = cm - eh \quad \bar{D} = de - cf, \\ \bar{E} &= dn - fk, \quad \bar{F} = fh - dm, \quad \bar{H} = hn - km,\end{aligned}$$

we obtain

$$\begin{aligned}\mu &= \bar{B}^2 + 4\bar{C}\bar{H} = 0, \quad \tilde{S} = -\bar{C}x^2 - \bar{B}xy + \bar{H}y^2, \\ H &= [\bar{C}(\bar{B} + 2\bar{F}) - \bar{B}\bar{D}]x + [\bar{B}(\bar{B} + \bar{F}) + 2\bar{H}(\bar{C} + \bar{D})]y \neq 0.\end{aligned}\tag{50}$$

We can see that if $\bar{B} = \bar{C} = \bar{H} = 0$ then $H = 0$ and the conditions from Theorem 5 are not valid. We shall prove, that in this case for system (49) $\bar{C} \neq 0$. Indeed, if we assume that $\bar{C} = 0$ from $\mu = 0$ and (50) we obtain $\bar{B} = 0$ and, hence, $\bar{H} = (hn - km) \neq 0$. Therefore, we get the linear homogeneous system:

$$\bar{C} = cm - eh = 0, \quad \bar{B} = cn - ek = 0,$$

with determinant (with respect to parameters c and e) is equal to $hn - km = \bar{H} \neq 0$. Thus, $c = e = 0$ and, hence, $\bar{D} = 0$ and again $H = 0$. This proves our assertion.

As we can observe from (50), invariant μ is the discriminant of the comitant \tilde{S} and since $\mu = 0$ we can write $\tilde{S} = (\alpha x + \beta y)^2 \neq 0$, where $\alpha \neq 0$ for $\bar{C} \neq 0$. Therefore, by applying the linear transformation

$$x_1 = x + \frac{\beta}{\alpha}y, \quad y_1 = y,$$

which keeps the singular points M_0 and M_1 , we obtain a new system of the same form (49), for which comitant \tilde{S} has the form $\tilde{S} = \gamma x_1^2$ (see [15], p. 26). Taking into account (50) this form of comitant K implies the relations $ek - cn = hn - km = 0$ and from $eh - cm \neq 0$ we get $n = k = 0$. This leads to the system

$$\frac{dx}{dt} = cx + dy - cx^2 + 2hxy, \quad \frac{dy}{dt} = ex + fy - ex^2 + 2mxy, \tag{51}$$

for which

$$\begin{aligned}\mu &= 0, \quad H = 2\bar{C}\bar{F}x \neq 0, \\ D &= -\frac{8}{27}\bar{D}^2\bar{F}^2(2\bar{C} - \bar{D})^2 < 0.\end{aligned}\tag{52}$$

It is easy to see that system (51) has singular points

$$M_0(0,0), \quad M_1(1,0), \quad M_2\left(\frac{\bar{D}}{2\bar{C}}, \frac{\bar{D}(2\bar{C} - \bar{D})}{4\bar{C}\bar{F}}\right)$$

and from [19] we obtain that for these points

$$\begin{aligned}\sigma^{(0)} &= c + f, \quad \sigma^{(1)} = f + 2m - c, \\ \sigma^{(2)} &= c + f + (m - c) \frac{\bar{D}}{C} + h \frac{\bar{D}(2\bar{C} - \bar{D})}{2\bar{C}\bar{F}},\end{aligned}\tag{53}$$

correspondingly. For system (51) we have:

$$C_1 = -24\bar{C}\bar{F}h\sigma^{(0)}\sigma^{(1)}\sigma^{(2)}, \quad C_4 = hP', \tag{54}$$

where P' is a polynomial in the coefficients of system (51), and, hence, if $C_1 \neq 0$ we get $\sigma^{(0)}\sigma^{(1)}\sigma^{(2)} \neq 0$ and by [19] system (51) has no center.

Let us assume that $C_1 = 0$. We shall consider two cases: $C_4 \neq 0$ and $C_4 = 0$.

1) If $C_4 \neq 0$ then according to (54) $h \neq 0$ and, hence, the condition $C_1 = 0$ implies that $\sigma^{(0)}\sigma^{(1)}\sigma^{(2)} = 0$. Without loss of generality we can assume that $\sigma^{(0)} = 0$, otherwise we can use a linear transformation, which will replace the points M_0 and M_1 in case $\sigma^{(1)} = 0$ or points M_0 and M_2 in case $\sigma^{(2)} = 0$. In both cases, after the corresponding change of parameters, we obtain the same system, but with $\sigma^{(0)} = 0$. Thus, from (53), we obtain $f = -c$ and for system (51) the following invariants can be calculated:

$$\begin{aligned}\bar{D}C_4 &= -\frac{1}{3}\bar{C}\bar{F}h\sigma^{(1)}\sigma^{(2)} = \frac{1}{3}\bar{D}I_{13}^{(0)}, \\ C_2 &= 3I_2^{(0)}C_4, \quad C_3 = -\frac{2}{3}I_6^{(0)}, \quad C_5 = 6I_3^{(0)}C_4, \\ C_6 &= 6(5I_3^{(0)} - 2I_4^{(0)})C_4, \quad C_7 = 2(13I_3^{(0)} - 10I_5^{(0)})C_4,\end{aligned}\tag{55}$$

where $I_j^{(0)}$ ($j = 2, 3, 4, 5, 6, 13$) are the values of the center affine invariants of Proposition 3, calculated for system (51) with singular point $M_0(0, 0)$. From $C_4 \neq 0$ and (54) we get $\sigma^{(1)}\sigma^{(2)} \neq 0$ and neither of the singular points M_1 or M_2 can be a center. In this case, from Proposition 3, relation $I_1 = c + f = 0$ and (55), we conclude that the singular point $M_0(0, 0)$ will be a center if and only if the following conditions hold:

$$C_1 = C_3 = C_5(C_6^2 + C_7^2) = 0, \quad C_2C_4 < 0.$$

However, we shall prove that in the case under consideration conditions $C_5 \neq 0$ and $C_6 = C_7 = 0$ can not be satisfied. Indeed, if we suppose the contrary, then from (55) it follows: $I_{13} \neq 0$, $I_2 < 0$, $I_1 = I_6 = 0$, $I_3 \neq 0$ and $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$. Hereby, as it was shown in [20, p.131], by applying a center affine transformation system (51) can be brought to the canonical system

$$\frac{dx}{dt} = y + qx^2 + xy, \quad \frac{dy}{dt} = -x - x^2 + 3qxy + 2y^2,$$

for which $D = 8q^2(q^2 + 1) > 0$ in virtue of $I_{13} = 125q(q^2 + 1)/8 \neq 0$. As we can observe this contradicts condition $D < 0$ of Theorem 5.

2) If the condition $C_4 = 0$ is satisfied from Lemma 1 (which are necessary for the existence of a center) conditions $C_1 = C_3 = 0$ imply that $h = 0$. Indeed, if

$h \neq 0$ as it was shown above, taking into account (54) and (55) we can assume, without loss of generality, that conditions $C_1 = C_4 = 0$ imply that $\sigma^{(0)} = \sigma^{(1)} = 0$. Hereby, from (53) it follows that $f = -c$, $m = c$ and for system (51) we obtain $C_3 = \frac{4}{3}ch(c^2 - eh) = 0$, however $H = -2c(c^2 - eh)(h + d)x \neq 0$.

Thus, $h = 0$, and for system (51), using (52), we obtain

$$H = 2cdm^2x \neq 0, \quad C_{10} = -cd^2m^3\sigma^{(0)}\sigma^{(1)}\sigma^{(2)}. \quad (56)$$

According to [19], for the existence of a center it is necessary that $C_{10} = 0$ and from $H \neq 0$ and (56), it follows that $\sigma^{(0)}\sigma^{(1)}\sigma^{(2)} = 0$. As it was mentioned above we can assume $\sigma^{(0)} = 0$. Therefore, the conditions $f = -c$ holds and for system (51) we obtain

$$\begin{aligned} C_3 &= \frac{2}{3}d(c-m)(c^2 + 2cm + de) = -\frac{2}{3}I_6^{(0)}, \quad I_1^{(0)} = I_{13}^{(0)} = 0, \\ C_{11} &= \frac{8}{3}d^2m^2(c-m)^2(c^2 + de) = \frac{4}{3}d^2m^2(c-m)^2I_2^{(0)}, \end{aligned} \quad (57)$$

where $I_j^{(0)}$ ($j = 1, 2, 6, 13$) are the values of the center affine invariants from Proposition 3, calculated for system (51) with $h = 0$ and $f = -c$.

If $C_{11} \neq 0$ we get from (57) that $SgnC_{11} = SgnI_2^{(0)}$ and from Proposition 3 and (57), we conclude that the singular point $M_0(0, 0)$ of system (51) will be a center if and only if $C_3 = 0$ and $C_{11} < 0$.

If condition $C_{11} = 0$ is satisfied, then from $\bar{D} = de + c^2 \neq 0$ and (57), it follows that $m = c$, so our system becomes

$$\frac{dx}{dt} = cx + dy - cx^2, \quad \frac{dy}{dt} = ex - cy - ex^2 + 2cxy, \quad (58)$$

the singular points of which are the following

$$M_0(0, 0), \quad M_1(1, 0), \quad M_2\left(\frac{c^2 + de}{2c^2}, \frac{d^2e^2 - c^4}{4c^3d}\right).$$

According to Proposition 3, we shall calculate the values of the invariants I_1 , I_2 , I_6 and I_{13} for system (58) as well as for others two with critical points M_1 and M_2 situated at the origin, respectively. Thus, we get the following expressions:

$$\begin{aligned} I_1^{(i)} &= I_6^{(i)} = I_{13}^{(i)} = 0 (i = 0, 1, 2), \\ I_2^{(0)} &= 2(c^2 + de) \neq 0, \quad I_2^{(1)} = 2(c^2 - de) \neq 0, \quad I_2^{(2)} = \frac{1}{2c^2}(de + c^2)(de - c^2), \end{aligned} \quad (59)$$

correspondingly. Since the condition $SgnI_2^{(2)} = -Sgn(I_2^{(0)}I_2^{(1)})$ holds, we can conclude, that among the quantities $I_2^{(i)}$ ($i = 0, 1, 2$) one and only one will be negative. Hence, in the case under consideration, one and only one of the singular points of system (58) is of the center type.

Since all possible cases were examined Theorem 5 is proved.

Theorem 6. For the existence of a center of system (1) with $\mu = 0$, $H \neq 0$, $D > 0$ (there are one simple real and two imaginary singular points) it is necessary and sufficient that the following conditions hold:

$$C_3 = C_9 = C_{10} = 0, \quad C_{11} < 0.$$

Moreover, the center is unique.

Proof. Let us assume that system (1) has one simple real and two imaginary singular points. By applying the affine transformation we can replace the real singular point by the origin. Hence, system (1) becomes

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy + lx^2 + 2mxy + ny^2, \end{aligned} \tag{60}$$

for which, by using the notations

$$\begin{aligned} \overset{*}{B} &= gn - lk, \quad \overset{*}{C} = cm - eh, \quad \overset{*}{D} = de - cf, \\ \overset{*}{E} &= dn - fk, \quad \overset{*}{F} = fh - dm, \quad \overset{*}{H} = hn - km, \\ \overset{*}{G} &= gm - hl, \quad \overset{*}{L} = dl - fg, \quad \overset{*}{N} = cn - ek \end{aligned}$$

we obtain

$$\begin{aligned} \mu &= \overset{*}{B}^2 - 4\overset{*}{G}\overset{*}{H} = 0, \quad \tilde{S} = \overset{*}{G}x^2 + \overset{*}{B}xy + \overset{*}{H}y^2, \\ H &= [\overset{*}{B}(\overset{*}{C} - \overset{*}{L}) - 2\overset{*}{G}(\overset{*}{F} + \overset{*}{N})]x + [2\overset{*}{H}(\overset{*}{C} - \overset{*}{L}) - \overset{*}{B}(\overset{*}{F} + \overset{*}{N})]y \neq 0. \end{aligned} \tag{61}$$

It is easy to see that from $\overset{*}{H}^2 + \overset{*}{G}^2 = 0$ it follows that $\overset{*}{B} = 0$ and, hence, $H = 0$, i.e. conditions of Theorem 6 are not valid. Therefore, $\overset{*}{H}^2 + \overset{*}{G}^2 \neq 0$ and we can consider $\overset{*}{G} \neq 0$ (changing the coordinate axes if it is necessary).

As one can easily obtain from (61), the invariant μ is the discriminant of the comitant \tilde{S} and in virtue of condition $\mu = 0$ we can write $\tilde{S} = (\alpha x + \beta y)^2 \neq 0$, where $\alpha \neq 0$ since $\overset{*}{C} \neq 0$. Therefore, applying the linear transformation

$$x_1 = x + \frac{\beta}{\alpha}y, \quad y_1 = y,$$

we obtain a new system of the same form (60), for which the comitant \tilde{S} has the form $\tilde{S} = \gamma x_1^2$ (see [15], p. 26). By (61) this form of comitant \tilde{S} implies the relations $\overset{*}{H} = hn - km = 0$, $\overset{*}{B} = gn - lk = 0$ and from $\overset{*}{G} = gm - hl \neq 0$ we get $n = k = 0$. This leads us to the system

$$\frac{dx}{dt} = cx + dy + gx^2 + 2hxy, \quad \frac{dy}{dt} = ex + fy + lx^2 + 2mxy, \tag{62}$$

for which

$$\begin{aligned} H &= 2\overset{*}{G}\overset{*}{F}x \neq 0, \quad D = -\frac{8}{27}\overset{*}{D}^2\overset{*}{F}^2Z > 0, \\ Z &= (\overset{*}{L} + 2\overset{*}{C})^2 + \overset{*}{F}\overset{*}{M}. \end{aligned} \tag{63}$$

From $D > 0$ it follows that $Z < 0$ and for system (62) we obtain

$$\begin{aligned} 2\overset{*}{G}\overset{*}{F}C_1 &= 3h\sigma^{(0)}(R^2 - ZI^2), \quad \sigma^{(0)} = c + f, \\ \overset{*}{F}C_4 &= \frac{h}{12}[Rh + I(dG - 2hC - 3gF - 4mF)], \quad C_4 = h^2\overset{*}{G}, \end{aligned} \tag{64}$$

where

$$\begin{aligned} R &= 4(c + f)\overset{*}{F}\overset{*}{G} + 2(g + m)\overset{*}{F}(\overset{*}{L} - 2\overset{*}{C}) + h(\overset{*}{L}^2 + 2\overset{*}{F}\overset{*}{M} + 2\overset{*}{D}\overset{*}{G}), \\ I &= 2(g + m)\overset{*}{F} + h\overset{*}{L}. \end{aligned}$$

There are two important cases to be examined: $C_9 \neq 0$ and $C_9 = 0$.

I. Let us assume first that condition $C_9 \neq 0$ hold. We shall prove the non-existence of a center for system (62).

From $C_9 \neq 0$ and (64) we get $h \neq 0$ and we can put $h = 1$ by changing the time if necessary. Therefore, we can consider, without loss of generality, that condition $g = 0$ holds, otherwise this can be obtained by applying the transformation $x_1 = x$, $y_1 = gx/2 + y$. Thus, we obtain the system

$$\frac{dx}{dt} = cx + dy + 2xy, \quad \frac{dy}{dt} = ex + fy + lx^2 + 2mxy, \tag{65}$$

for which in accordance with Proposition 3, the condition $I_1 = c + f = 0$ gives $f = -c$ and then

$$I_6 = cd़l + 2cdm^2 + 2ce - d^2lm, \quad Z = 4c(cm^2 - 2cl - dlm - 2em) + (dl - 2e)^2 < 0.$$

As condition $Z < 0$ implies $c \neq 0$ we shall introduce a new parameter u by setting $l = 2cu$. Hereby, taking into consideration Proposition 3, the necessary for the existence of the center condition $I_6 = 0$ implies that $e = d^2mu - cdu - dm^2$ and for system (65) the following expressions can be obtained:

$$\begin{aligned} Z &= 4(c - dm)[c(2du - m)^2 - 4c^2u + dm^2(3m - 2du)] + 4d^2m^2(du - 2m)^2, \\ H &= -4cu(c + dm)x \neq 0, \quad I_1 = I_6 = 0, \quad I_{13} = 2u(c - dm)^2, \\ 2I_2 &= (2c - d^2u)^2 - d^2(2m - du)^2, \quad I_3 = 2c(2m - du), \\ 5I_3 - 2I_4 &= 2(2m - du)(4c + dm), \\ 13I_3 - 10I_5 &= 2(4c + dm)(3du + 4m) + 4dm(3m - 4du). \end{aligned} \tag{66}$$

Since $H \neq 0$, $Z < 0$ and $I_2 < 0$ from (66) we get $I_3I_{13} \neq 0$ and, hence, from Proposition 3, for the existence of a center at the origin of coordinates for system (65) it is necessary that conditions $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$ be satisfied.

However, as it is easily seen from (66) the condition $5I_3 - 2I_4 = 0$ leads to the condition $c = -dm/4$, and, hence,

$$13I_3 - 10I_5 = 4dm(3m - 4du) = 0, \quad 8I_2 = 5d^2m(4du - 3m) < 0.$$

The obtained contradiction proves our assertion.

II. Let conditions $C_9 = 0$ be satisfied, i.e. $h = 0$. In this case for system (62) we obtain $H = 2dgm^2x \neq 0$ and, hence, the conditions $d = 1$ and $c = 0$ can be considered to be satisfied. Indeed, by $H \neq 0$ it results $d \neq 0$ and by applying a change of the scale one can obtain $d = 1$. Hereby, after the transformation $x_1 = x$, $y_1 = cx + y$ system (62) will be transformed into the system

$$\frac{dx}{dt} = y + gx^2, \quad \frac{dy}{dt} = ex + fy + lx^2 + 2mxy, \quad (67)$$

for which

$$\begin{aligned} H &= 2gm^2x \neq 0, \quad Z = (fg - l)^2 + 8egm < 0, \\ 3C_3 &= 2[l(g + m) + fg(2m - g)], \\ C_{10} &= fm^2[f(lm - g^2f + gl)] - 2e(g + m)^2, \\ 3C_{11} &= 8m^2(g + m)[e(g + m) - lf] - 4f^2gm^2(m - 2g), \\ I_1 &= f, \quad I_2 = 2e + f^2, \quad I_6 = -(g + m)(l + fm), \quad I_{13} = 0. \end{aligned} \quad (68)$$

To prove the existence of a center in the case we discussing, according to Proposition 3 it remains to prove the equivalence of the sequence of the conditions $C_3 = C_{10} = 0$, $C_{11} < 0$ with the following one: $I_1 = I_6 = 0$, $I_2 < 0$.

If conditions $C_3 = C_{10} = 0$ and $C_{11} < 0$ are satisfied then $g + m \neq 0$, otherwise $m = -g$ and from (68) we obtain $C_{11} = 4f^2g^4 \geq 0$. Therefore some new parameter w can be introduced, namely: $f = (g + m)w$. It is easily seen from (68) that condition $C_3 = 0$ gives $l = gw(g - 2m)$ and for system (67) we have

$$C_{10} = -2wm^2(g + m)^3(gmw^2 + e) = 0, \quad Z = 9g^2m^2w^2 + 8egm < 0.$$

We note that $e \neq -gmw^2$, otherwise $Z = g^2m^2w^2 \geq 0$. Hence, the condition $C_{10} = 0$ implies that $w = 0$, i.e. $f = l = 0$ and, according to (68), from $3C_{11} = 8em^2(g + m)^2 < 0$ we get $e < 0$.

Thus, we have demonstrated, that the conditions $C_3 = C_{10} = 0$ and $C_{11} < 0$ for system (67) with $Z < 0$ are equivalent to $f = l = 0$, $e < 0$. On the other hand, from (67) it is not difficult to observe, that from $I_1 = I_6 = 0$, $I_2 < 0$ and $Z < 0$ it also follows that $f = l = 0$, $e < 0$. This fact proves our assertion.

As all possible cases were examined, Theorem 6 is proved.

In accordance with [21] it occurs

Theorem 7. *Quadratic system (1) with $\mu = D = 0$, $R \neq 0$, $P \neq 0$ (so there are one simple and one double singular points) has not a critical point of the center type.*

We shall include hear an independent **proof** of this assertion.

From Proposition 2 and $\mu = D = 0$ the conditions $R \neq 0$ and $P \neq 0$ are equivalent to $H \neq 0$ and $G^2 - 6HF \neq 0$, respectively. We recall that in this case the system (1) has one simple and one double singular points situated on its phase plane. In order to find out the corresponding canonical form of system (1) as it was indicated in the proof of Theorem 5, by using an affine transformation we can replace the two real singular points by the points $M_0(0, 0)$ and $M_1(1, 0)$, respectively. Moreover, the system (1) with conditions $\mu = 0$, $H \neq 0$ can be transformed by the linear transformation into the system

$$\frac{dx}{dt} = cx + dy - cx^2 + 2hxy, \quad \frac{dy}{dt} = ex + fy - ex^2 + 2mxy, \quad (69)$$

for which, by using the notations

$$\bar{C} = cm - eh, \quad \bar{D} = de - cf, \quad \bar{F} = fh - dm,$$

one can obtain

$$\mu = 0, \quad H = 2\bar{C}\bar{F}x \neq 0, \quad D = -\frac{8}{27}\bar{D}^2\bar{H}^2(2\bar{C} - \bar{D})^2.$$

As it was shown above system (69) has singular points

$$M_0(0, 0), \quad M_1(1, 0), \quad M_2\left(\frac{\bar{D}}{2\bar{C}}, \frac{\bar{D}(2\bar{C} - \bar{D})}{4\bar{C}\bar{F}}\right).$$

Hereby, we can deduce that by $D = 0$, the point M_2 will coincide with M_0 for $\bar{D} = 0$ and with point M_1 for $\bar{D} = 2\bar{C}$. Without loss of generality we can assume that the condition $D = 0$ implies $\bar{D} = de - cf = 0$ and singular point M_0 becomes degenerated (the points can be replaced if it is necessary). From $\bar{C} \neq 0$ we get $c^2 + e^2 \neq 0$ and we can set $d = cu$, $f = eu$. Thus system (69) becomes the system

$$\begin{aligned} \frac{dx}{dt} &= cx + cuy - cx^2 + 2hxy, \\ \frac{dy}{dt} &= ex + euy - ex^2 + 2mxy, \end{aligned}$$

which, after placing of the origin at the point $M_1(1, 0)$, gets into the form

$$\begin{aligned} \frac{dx}{dt} &= -cx + (cu + 2h)y - cx^2 + 2hxy, \\ \frac{dy}{dt} &= -ex + (eu + 2m)y - ex^2 + 2mxy. \end{aligned} \quad (70)$$

According to Proposition 3 for the existence of a center it is necessary $I_1 = 0$. Therefore, for system (70) condition $I_1 = eu - c + 2m = 0$ yields $c = eu + 2m$ and then, for this system we obtain

$$\begin{aligned} I_6 &= 2u(emu + 2m^2 - eh)(eh + 2e^2u^2 + 6uem + 4m^2) = 0, \\ I_2 &= 4(emu + 2m^2 - eh) < 0. \end{aligned} \quad (71)$$

From $I_2 < 0$, $I_6 = 0$ and (71) we get $eh + 2e^2u^2 + 6uem + 4m^2 = 0$. Since $H \neq 0$, the condition $e^2 + m^2 \neq 0$ holds. Then $e \neq 0$ and, by changing the time we can obtain $e = 1$. Therefore we have $h = -2(u^2 + 3um + 2m^2)$ and for the system (70) we obtain

$$\begin{aligned} I_1 &= I_6 = 0, \quad I_2 = 4(2u + 3m)(u + 2m) \\ I_3 &= (5u + 12m)(u + m)(u + 2m)^2, \\ 5I_3 - 2I_4 &= (5u + 12m)(7u + 12m)(u + m)(u + 2m), \\ 13I_3 - 10I_5 &= -(u + 2m)(5u^3 + 9mu^2 + 32m^2u + 48m^3), \\ H &= -2u(2u + 3m)^2(u + 2m)^2x \neq 0. \end{aligned} \tag{72}$$

It is easy to see that in virtue of $H \neq 0$ and (72) the condition $I_3 = 0$ yields $5u + 12m = 0$. However, in this case $I_2 = 72m^2/25 \geq 0$ and in accordance with Proposition 3 system (70) has not a center.

On the other hand, it is easy to observe, that conditions $5I_3 - 2I_4 = 13I_3 - 10I_5 = 0$ can not be satisfied simultaneously, because neither of the quantities $u = -12m/5$ nor $u = -12m/7$ will satisfy the relation $5u^3 + 9mu^2 + 32m^2u + 48m^3 = 0$.

Thus, from Proposition 3, we can conclude that the affirmation of Theorem 7 is valid.

§3. System with total multiplicity $m_f \leq 2$

In this section we shall determine the conditions for the existence of a center in the case where the total multiplicity m_f is less than or equal to two. But for a complete solution of the center problem, besides the invariants (3) and those from Table 1 we also need the following elements of the minimal polynomial basis of the center-affine comitants [15]:

$$\begin{aligned} I_{17} &= a^\alpha a_{\alpha\beta}^\beta, \quad I_{20} = a^\alpha a^\beta a_\alpha^\gamma a_{\alpha\beta}^\delta \varepsilon_{\delta\gamma}, \\ K_1 &= a_{\alpha\beta}^\alpha x^\beta, \quad J_2 = I_1(I_2 - I_1^2) + 4I_1 I_{17} - 4I_{20}. \end{aligned}$$

Theorem 8. System (1) with conditions $\mu = R = 0$, $P \neq 0$, $U > 0$ (there are two simple singular points) has one center if and only if one of the following three sequences of conditions holds:

- (i) $C_2 C_4 < 0$, $C_1 = C_3 = C_5 = 0$;
- (ii) $C_1 = C_3 = C_4 = 0$, $C_8 > 0$;
- (iii) $\tilde{S} = 0$, $K_1 = 0$, $I_1 = 0$;

and it has two centers if and only if the following conditions hold

- (iv) $C_1 = C_3 = C_4 = 0$, $C_8 < 0$, $C_9 > 0$.

Proof. Let us assume that conditions $\mu = R = 0$, $P \neq 0$, $U > 0$ are valid for system (1). From Proposition 2, we can easily see that these conditions are

equivalent to the following ones: $\mu = H = 0$, $G \neq 0$, $U > 0$. Since in this case the system (1) has two real simple singular points, by using an affine transformation we can replace them by the points $M_0(0, 0)$ and $M_1(1, 0)$, respectively. Thus, we obtain the system

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2mxy + ny^2,\end{aligned}\tag{73}$$

for which, by using the notations

$$\begin{aligned}\bar{B} &= cn - ek, \quad \bar{C} = cm - eh \quad \bar{D} = de - cf, \\ \bar{E} &= dn - fk, \quad \bar{F} = fh - dm, \quad \bar{H} = hn - km\end{aligned}$$

we obtain

$$\begin{aligned}\mu &= \bar{B}^2 + 4\bar{C}\bar{H} = 0, \quad \tilde{S} = -\bar{C}x^2 - \bar{B}xy + \bar{H}y^2, \\ H &= [\bar{C}(\bar{B} + 2\bar{F}) - \bar{B}\bar{D}]x + [\bar{B}(\bar{B} + \bar{F}) + 2\bar{H}(\bar{C} + \bar{D})]y \neq 0.\end{aligned}\tag{74}$$

It will be convenient to examine the cases: $\tilde{S} \neq 0$ and $\tilde{S} = 0$.

A. If $\tilde{S} \neq 0$, it follows from the proof of Theorem 5 and $\mu = 0$, that there exists a center affine transformation that brings system (73) to the same form but with the additional conditions: $k = n = 0$. Thereby for this system we obtain

$$\mu = 0, \quad S = (cm - eh)x^2 \neq 0, \quad H = 2(cm - eh)(fh - dm)x = 0.$$

Since $d^2 + f^2 \neq 0$ (otherwise system (73) with $k = n = 0$ becomes degenerated, and then $G = 0$) by $H = 0$, $\tilde{S} \neq 0$, we may assume, without loss of generality, that conditions $h = du$, $m = fu$ hold. Thus, in the case $\tilde{S} \neq 0$, we have obtained the following canonical form

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2duxy, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2fuxy,\end{aligned}\tag{75}$$

for which

$$\tilde{S} = u(de - cf)x^2 \neq 0, \quad G = (cf - de)^2(2u + 1)x^2 \neq 0.$$

On the other hand we shall simultaneously consider the system

$$\begin{aligned}\frac{dx}{dt} &= -cx + d(2u + 1)y - cx^2 + 2duxy, \\ \frac{dy}{dt} &= -ex + f(2u + 1)y - ex^2 + 2fuxy,\end{aligned}\tag{76}$$

which is obtained from (75) by replacing the origin at the singular point $M_1(1, 0)$.

For system (75), as well as for system (76), the values of the following affine invariants can be calculated:

$$\begin{aligned}C_1 &= 12d^2u^2(2u + 1)(cf - de)^2\sigma^{(0)}\sigma^{(1)}, \\ C_8 &= -\frac{4}{3}d^2u^2(2u + 1)(de - cf)^2.\end{aligned}\tag{77}$$

We note, that the quantities $\sigma^{(0)} = c + f$ and $\sigma^{(1)} = -c + f + 2fu$ correspond to the singular points M_0 and M_1 , respectively.

I. Let us consider at first that condition $C_8 \neq 0$ is valid. According to Lemma 1 from $GC_8 \neq 0$ and (77) the condition $C_1 = 0$ (which is necessary for the existence of a center) yields $\sigma^{(0)}\sigma^{(1)} = 0$. Without loss of generality one can consider $\sigma^{(0)} = 0$ otherwise the linear transformation which replace the points M_0 and M_1 and keeps the canonical form of system (75) can be applied.

Thus, $f = -c$ and for system (75) we obtain

$$\begin{aligned} C_3 &= \frac{2}{3}cd(2u+1)(de+c^2)(1-u), \\ C_4 &= -\frac{1}{3}cd^2u^2(2u+1)(de+c^2)(u+1). \end{aligned} \quad (78)$$

1) If $C_4 \neq 0$ we shall prove that there exists no center for system (75). Indeed, by (78) and Lemma 1 the second necessary for the existence of a center condition $C_3 = 0$ by virtue of $C_4 \neq 0$ implies $u = 1$. Therefore, for system (75) in view of Proposition 3, we can calculate

$$\begin{aligned} I_1^{(0)} &= I_6^{(0)} = 0, \quad I_3^{(0)} = d(de - 8c^2) = \frac{1}{3}(5I_3^{(0)} - 2I_4^{(0)}), \\ I_2^{(0)} &= 2(c^2 + de), \quad I_{13}^{(0)} = -6cd^2(c^2 + de). \end{aligned}$$

Since $I_{13}^{(0)} \neq 0$ for $C_4 \neq 0$, according to Proposition 3, the singular point M_0 will be a center only if $I_3^{(0)} = 0$, but the condition $de = 8c^2$ yields $I_2^{(0)} = 18c^2 > 0$. Hence, from $\sigma^{(1)} = -4c \neq 0$ we deduce that for $C_4 \neq 0$ and $\tilde{S} \neq 0$ the system (73) has no a center on its phase plane.

2) Let us now assume that $C_4 = 0$. From $C_3 = C_4 = 0$ and (78) we get $c = 0$ and for system (75) we obtain

$$\begin{aligned} I_1^{(0)} &= I_6^{(0)} = I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2de, \\ C_8 &= \frac{4}{3}d^2e^2u^2(2u+1), \quad C_9 = d^3eu^3, \end{aligned}$$

and for system (76), in the same way we can calculate

$$I_1^{(1)} = I_6^{(1)} = I_{13}^{(1)} = 0, \quad I_2^{(1)} = -2de(2u+1).$$

Since the following relations

$$SgnC_8 = -Sgn(I_2^{(0)}I_2^{(1)}), \quad SgnC_9 = -Sgn(I_2^{(0)} + I_2^{(1)})$$

hold, from Proposition 3, we conclude that if $\tilde{S}C_8 \neq 0$, then $C_1 = C_3 = C_4 = 0$ and the system (73) has one center if $C_8 > 0$, and it has two centers if $C_8 < 0$, $C_9 > 0$.

II. Let condition $C_8 = 0$ be satisfied. Taking into consideration condition $\tilde{S}G \neq 0$ and (77) condition $C_8 = 0$ yields $d = 0$ and then, for systems (75) and (76), we get $I_2^{(0)} = c^2 + f^2 \geq 0$ and $I_2^{(1)} = c^2 + f^2(2u+1)^2$, respectively. By virtue of Proposition 3 there exists no center on the phase plane of system (73).

B. Let us assume that condition $\tilde{S} = 0$ holds. From (74) it follows that $\bar{C} = cm - eh = 0$, $\bar{B} = cn - ek = 0$, $\bar{H} = hn - km = 0$. It is easy to see that condition $c^2 + e^2 \neq 0$ holds for system (74), otherwise this system becomes degenerated and then $G = 0$. Therefore, without loss of generality, one can be assumed that relations $h = cu$, $m = eu$ and $k = cv$, $n = ev$ are valid, where u and v are some new independent parameters.

Thus, system (73) will be transformed into the system

$$\begin{aligned}\frac{dx}{dt} &= cx + dy - cx^2 + 2cuxy + cvy^2, \\ \frac{dy}{dt} &= ex + fy - ex^2 + 2euxy + evy^2,\end{aligned}\tag{79}$$

for which

$$\mu = 0, \tilde{S} = 0, G = -(cf - de)^2(-x^2 + 2uxy + vy^2) \neq 0, K_1 = (eu - c)x + (cu + ev)y.$$

By translating the origin of coordinate at the singular point $M_1(1, 0)$ of system (79) we obtain the system

$$\begin{aligned}\frac{dx}{dt} &= -cx + (2cu + d)y - cx^2 + 2cuxy + cvy^2, \\ \frac{dy}{dt} &= -ex + (2eu + f)y - ex^2 + 2euxy + evy^2.\end{aligned}\tag{80}$$

For system (79) as well as for system (80) one can find out the values of the following affine invariants:

$$\begin{aligned}C_1 &= 12(cf - de)^2(v + u^2)(2ceu - c^2 + e^2v)\sigma^{(0)}\sigma^{(1)}, \\ C_4 &= \frac{1}{3}(cf - de)(v + u^2)(2ceu - c^2 + e^2v)(c - ue), \\ C_8 &= \frac{4}{3}(cf - de)^2(v + u^2)(2ceu - c^2 + e^2v),\end{aligned}\tag{81}$$

where $\sigma^{(0)} = c + f$ and $\sigma^{(1)} = -c + f + 2eu$.

I. If $C_8 \neq 0$ in accordance with Lemma 1, the condition $C_1 = 0$ $G \neq 0$ and (81) imply that $\sigma^{(0)}\sigma^{(1)} = 0$. By the same reasoning we obtain $\sigma^{(0)} = 0$. Hence $f = -c$, and for the system (79) we obtain

$$\begin{aligned}C_3 &= -\frac{1}{3}(c^2 + de)(2cu + d + ev)\sigma^{(1)}, \\ C_4 &= \frac{1}{6}(c^2 + de)(v + u^2)(2ceu - c^2 + e^2v)\sigma^{(1)},\end{aligned}\tag{82}$$

1) Let us assume that the condition $C_4 \neq 0$ holds. Hence, $\sigma^{(1)} \neq 0$ and singular point $M_1(1, 0)$ of system (79) (i.e. point $(0, 0)$ of system (80)) is not a center. Hereby, $C_3 = 0$ and (82) yield $d = -2cu - ev$ and we obtain:

$$C_2 = -6(2ceu - c^2 + e^2v)C_4, \quad C_5 = 0.$$

On the other hand, by Proposition 3 for system (79) we can calculate

$$I_1^{(0)} = I_3^{(0)} = I_6^{(0)} = 0, \quad I_2^{(0)} = 2(c^2 - 2ceu - e^2v)$$

and, as it can be easily seen, the following relation holds: $Sgn(C_2C_4) = SgnI_2^{(0)}$. Hence, in accordance with Proposition 3 if $C_2C_4 < 0$ there exists one center on the phase plane of system (79).

2) If condition $C_4 = 0$ holds then since $C_8 \neq 0$, $\sigma^{(0)} = 0$ and (81) from (82) we get $\sigma^{(1)} = -c + f + 2eu = 0$ and from $\sigma^{(0)} = c + f = 0$ we obtain $\sigma^{(1)} = 2(eu - c) = 0$. Thus, $c = eu$ and for system (79), as well as for system (80), we obtain

$$\begin{aligned} I_1^{(0)} &= I_6^{(0)} = I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2(de + eu^2), \\ I_1^{(1)} &= I_6^{(1)} = I_{13}^{(1)} = 0, \quad I_2^{(1)} = -2(de + eu^2), \end{aligned}$$

respectively. Since $I_2^{(0)}I_2^{(1)} < 0$ by Proposition 3 for system (73) either singular point M_0 or M_1 is of the center type.

It is important to underline, that in the case when $\tilde{S} = 0$, $C_8 \neq 0$ and $C_4 = 0$ we have obtained $C_8 = \frac{4}{3}e^4(d + eu^2)^2(v + u^2)^2 > 0$ and, hence, these condition can be united with the same one of the case $S \neq 0$ excluding conditions $\tilde{S} \neq 0$ and $\tilde{S} = 0$, and namely:

If $C_8 \neq 0$ and $C_4 = 0$ system (73) has one center for $C_1 = C_3 = 0$, $C_8 > 0$ and has two centers for $C_1 = C_3 = 0$, $C_8 < 0$, $C_9 > 0$.

II. Let us assume now that condition $C_8 = 0$ hold, i.e. in accordance with (81) one have $(v + u^2)(2ceu - c^2 + e^2v) = 0$. We shall prove that the condition $C_3 = 0$ (which is necessary for the existence of a center) implies $v + u^2 = 0$. Indeed, if we assume that $v + u^2 \neq 0$, by (81), condition $C_8 = 0$ yields $2ceu - c^2 + e^2v = 0$. We have already noted, that condition $c^2 + e^2 \neq 0$ is valid for system (80) and, hence, $e \neq 0$ because of $2ceu - c^2 + e^2v = 0$. Therefore, one can consider $e = 1$ (by changing the time if it is necessary) and $v = c^2 - 2cu$. Hereby, for system (79) we obtain

$$C_3 = \frac{2}{3}(cf - d)^2(c - u), \quad G = (cf - d)^2(x - cy)[x + (c - 2u)y] \neq 0.$$

Hence, by $G \neq 0$, the condition $C_3 = 0$ yields $c = u$, and the relation $v = c^2 - 2cu$ becomes $v = -u^2$.

Thus, conditions $C_8 = C_3 = 0$ imply $v = -u^2$, and for system (79) we can calculate:

$$C_3 = \frac{2}{3}(cf - de)(c - ue)(fu - cu - d - eu^2), \quad K_1 = (eu - c)(x - uy).$$

1) If $K_1 \neq 0$, from $C_3 = 0$ it follows that $d = fu - cu - eu^2$ and for system (79) as well as for system (80), we obtain

$$I_2^{(0)} = I_2^{(1)} = (c - eu)^2 + (f + eu)^2 \geq 0.$$

Hence, according to Proposition 3 for $K_1 \neq 0$ there exists no center for the system (79).

1) If $K_1 = 0$ then $c = eu$ (this implies $C_3 = 0$) and for the systems (79) and (80), respectively, we can obtain the following values of the invariants

$$\begin{aligned} I_1^{(0)} &= f + eu, \quad I_6^{(0)} = I_{13}^{(0)} = 0, \quad I_2^{(0)} = 2e(d - fu) + (f + eu)^2, \\ I_1^{(1)} &= f + eu, \quad I_6^{(1)} = I_{13}^{(1)} = 0, \quad I_2^{(1)} = -2e(d - fu) + (f + eu)^2. \end{aligned}$$

As it can be easily seen from $I_1^{(0)} = 0$ we get $I_1^{(1)} = 0$ and $I_2^{(0)}I_2^{(1)} < 0$. Since for $K_1 = 0$ the condition $I_1 = 0$ becomes an affine invariant one, we can deduce, that in case of $K_1 = 0$ the system (79) has one singular point of the center type if and only if $I_1 = 0$.

It remains to underline the following two moments.

The 1st: For system (79) from $K_1 = (eu - c)x + (cu + ev)y = 0$ it follows that $c = eu$, $v = -u^2$ and, hence, $C_3 = C_8 = 0$, i.e. these last conditions can be excluded from the respective consequence $\tilde{S} = 0$, $C_8 = C_3 = 0$, $K_1 = 0$, $I_1 = 0$.

The 2nd: If $\tilde{S} \neq 0$ the conditions $C_2C_4 < 0$, $C_1 = C_3 = C_5 = 0$ are not compatible. Indeed, as it was indicated above (see p. A, I, 1)), if $C_4 \neq 0$ the conditions $C_1 = C_3 = 0$ yields $f = -c$, $u = 1$ and then for system (75) one can be get out:

$$C_4 = -2cd^2(c^2 + de) \neq 0, \quad C_2 = -12cd^2(c^2 + de)^2, \quad C_5 = 6d(de - 8c^2)C_4.$$

Therefore, condition $C_5 = 0$ yields $de = c^2$ and, hence, $C_2C_4 = 216c^8d^4 > 0$. This proves our assertion and we can exclude condition $\tilde{S} = 0$ from the indicated above sequence of conditions.

As all cases were examined, Theorem 8 is proved.

Theorem 9. *For the existence of a center of system (1) with $\mu = H = G = 0$, $F \neq 0$ (there are one simple real singular point) it is necessary and sufficient that one of the following two sequences of conditions holds:*

- (i) $\tilde{N} \neq 0$, $C_3 = C_{10} = 0$, $C_{11} < 0$;
- (ii) $\tilde{N} = 0$, $J_2 = 0$, $J_1 > 0$.

Proof. Let us consider that conditions $\mu = R = P = 0$ and $U \neq 0$ are valid for system (1). Taking into consideration Proposition 2 one can easily be seen that these conditions are equivalent to the following ones: $\mu = H = G = 0$, $F \neq 0$. Since in this case system (1) has one real simple singular point we can consider it situated at the origin of coordinates, i.e. system (1) will be of the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^2 + 2hxy + ky^2, \\ \frac{dy}{dt} &= ex + fy + lx^2 + 2mxy + ny^2, \end{aligned} \tag{83}$$

for which,

$$\begin{aligned} \mu &= (gn - kl)^2 - 4(gm - hl)(hn - km) = 0, \\ \tilde{S} &= (gm - hl)x^2 + (gn - kl)xy + (hn - km)y^2. \end{aligned} \tag{84}$$

We shall examine two cases: $\tilde{S} \neq 0$ and $\tilde{S} = 0$.

A. If $\tilde{S} \neq 0$ we will prove that point M_0 of system (83) is not of the center type. Indeed, the invariant μ is the discriminant of the binary form S and, hence, the comitant S can be represented in the form $S = (\alpha x + \beta y)^2$. Since $S \neq 0$ we can assume $\alpha \neq 0$, otherwise replace x and y . Hereby, applying the linear transformation $x_1 = \alpha x + \beta y$, $y_1 = y$ we obtain system (83) for which the following conditions hold:

$$gn - kl = hn - km = 0, \quad gm - hl \neq 0.$$

As it can be easily seen, these conditions yield $k = n = 0$ and for system (83) we obtain:

$$S = (gm - hl)x^2 \neq 0, \quad H = 2(gm - hl)(dm - fh)x = 0.$$

Since $h^2 + m^2 \neq 0$ for $S \neq 0$ from $H = 0$, without loss of generality, one can put $d = 2hu$, $f = 2mu$, where u is a new parameter (a constant factor 2 is introduced for computational considerations). This takes us to the system

$$\begin{aligned} \frac{dx}{dt} &= cx + 2huy + gx^2 + 2hxy, \\ \frac{dy}{dt} &= ex + 2muy + lx^2 + 2mxy, \end{aligned} \tag{85}$$

for which,

$$\begin{aligned} \mu &= H = 0, \quad S = (gm - hl)x^2 \neq 0, \\ G &= 4u(gm - hl)[u(gm - hl) + (eh - cm)]x^2 = 0. \end{aligned} \tag{86}$$

1) If $h \neq 0$ one can consider $h = 1$ (otherwise a change of scale can be done) and, hence, condition $G = 0$ yields $e = cm - gmu + lu$. Hereby, from Proposition 3, for system (85) we obtain $I_1 = c + 2mu = 0$ and from this the following values of the comitants can be obtained:

$$F = 2u^3(g + 2m)(gm - l)^3x^3 \neq 0, \quad I_6 = 6u^2(g + 2m)(gm - l).$$

Therefore, condition $F \neq 0$ yields $I_6 \neq 0$ and, from Proposition 3, the singular point $M_0(0, 0)$ of system (85) is not a center.

2) If the condition $h = 0$ is satisfied, from $G = 0$, $S \neq 0$ and (80) it follows that $c = gu$. Hence, for system (85) we obtain $I_2 = u^2(g^2 + 4m^2) \geq 0$ and again from Proposition 3, there is no center on the phase plane of system (85).

B. Let us assume now that condition $\tilde{S} = 0$ holds. From (84) we get $gm - hl = gn - kl = hn - km = 0$ and, hence, the homogeneous quadratic parts of system (83) are proportional. Without any loss of generality we can assume $g = h = k = 0$, otherwise this can be obtained by applying a linear transformation.

Thus, system (83) can be transformed into the system

$$\frac{dx}{dt} = cx + dy, \quad \frac{dy}{dt} = ex + fy + lx^2 + 2mxy + ny^2, \tag{87}$$

for which

$$\begin{aligned} \mu &= H = 0, \quad G = (c^2n - 2cdm + d^2l)(lx^2 + 2mxy + ny^2) = 0, \\ F &= (cf - de)(cx + dy)(lx^2 + 2mxy + ny^2) \neq 0, \quad \tilde{N} = (m^2 - ln)x^2. \end{aligned} \tag{88}$$

1) Let us assume that condition $\tilde{N} \neq 0$ holds. Hereby, we shall prove that the singular point $M_0(0, 0)$ of the system (87) will be of the center type if and only if conditions $C_3 = C_{10} = 0$, $C_{11} < 0$ are satisfied.

We will show first that these conditions imply $d \neq 0$, because one can easily see that, for $d = 0$, system (87) has not a center at the origin. Indeed, if we assume $d = 0$ then, from (88) $G = 0$ and $F \neq 0$ it results $n = 0$ and for system (87) it follows at once $C_{11} = 0$.

Thus, $d \neq 0$ and one can assume $d = 1$, whence it follows from $G = 0$ and (88) that condition $l = 2cm - c^2n$ holds. Hereby, for system (87) we have

$$\begin{aligned} C_3 &= \frac{2}{3}n(m - cn)(cf - e), \quad \tilde{N} = (m - cn)^2x^2 \neq 0, \\ F &= (cf - e)(cx + y)(lx^2 + 2mxy + ny^2) \neq 0. \end{aligned}$$

According to Lemma 1, in order to have a center on the phase plane for the system (87) is necessary that $C_3 = 0$, and by virtue of $G\tilde{N} \neq 0$ it follows at once that $n = 0$. Thus, system (87) becomes

$$\frac{dx}{dt} = cx + y, \quad \frac{dy}{dt} = ex + fy + 2cmx^2 + 2mxy, \quad (89)$$

for which

$$\begin{aligned} C_{10} &= 2m^4(c + f)(cf - e), \quad C_{11} = \frac{8}{3}m^4(e - cf), \quad I_1 = (c + f), \\ I_2 &= (c + f)^2 + 2(e - cf), \quad I_6 = -m^2(c + f), \quad I_{13} = 0. \end{aligned} \quad (90)$$

Taking into consideration Proposition 3 and (90), we can deduce that the singular point $M_0(0, 0)$ of system (89) will be a center if and only if the conditions $I_1 = 0$ and $I_2 < 0$ are valid. By (90) and $F\tilde{N} \neq 0$ these conditions are equivalent to $C_{10} = 0$ and $C_{11} < 0$, respectively. This proves our assertion.

2) Let us now assume that $\tilde{N} = 0$, i.e. according to (88) the condition $m^2 - ln = 0$ is satisfied for system (87).

a) If $d \neq 0$ then by the same reason given above, we can put $d = 1$ and conditions $G = \tilde{N} = 0$ from (88) yield $m = cn$ and $l = c^2n$. Thus, the system (87) becomes

$$\frac{dx}{dt} = cx + y, \quad \frac{dy}{dt} = ex + fy + c^2nx^2 + 2cnxy + ny^2, \quad (91)$$

for which

$$\begin{aligned} \mu &= H = G = 0, \quad F = n(cf - e)(cx + y)^3 \neq 0, \\ J_2 &= 2(c + f)(e - cf), \quad J_1 = cf - e, \\ I_1 &= c + f, \quad I_2 = (c + f)^2 + 2(e - cf), \quad I_6 = n^2(c + f)(e - cf). \end{aligned} \quad (92)$$

By Proposition 3 and (92), we can conclude that the singular point $M_0(0, 0)$ of system (91) will be a center if and only if conditions $I_1 = 0$ and $I_2 < 0$ are valid. Hereby, we can easily see from (92) that $F \neq 0$ and these conditions are equivalent to the following ones: $J_2 = 0$, $J_1 > 0$.

b) In case $d = 0$, it remain to be shown that conditions $G = \tilde{N} = J_2 = 0$, and $J_1 > 0$ are not compatible. Indeed, $d = 0$ and $G = \tilde{N} = 0$, imply $n = m = 0$ and then it follows at once that $J_1 = cf$, $J_2 = -2cf(c + f)$. Hereby it is not difficult to see, that the condition $J_2 = 0$ yields $J_1 \leq 0$ and this proves our assertion.

Theorem 9 is proved.

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*Institute of Mathematics,
Academy of Science of Moldova
5 Academiei Str, Chishinău,
MD-20028, Moldova
Phone: (373-2) 727059
Fax: (373-2) 738027
E-mail: 15vulpe@mathem.moldova.su*

*443 Eucaliptus Drive,
Redlands, CA,
92373, U.S.A.
E-mail:
avoldman@zzyzx.math.csusb.edu*